AN ENTANGLED TALE OF QUANTUM ENTANGLEMENT VERSION 1.5

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ABSTRACT. These lecture notes give an overview from the perspective of Lie group theory of some of the recent advances in the rapidly expanding research area of quantum entanglement.

This paper is a written version of the last of eight one hour lectures given in the American Mathematical Society (AMS) Short Course on Quantum Computation held in conjunction with the Annual Meeting of the AMS in Washington, DC, USA in January 2000.

More information about the AMS Short Course can be found at the website: http://www.csee.umbc.edu/~lomonaco/ams/Announce.html

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1. Introduction

These lecture notes were written for the American Mathematical Society (AMS) Short Course on Quantum Computation held 17-18 January 2000 in conjunction with the Annual Meeting of the AMS in Washington, DC in January 2000.

The objective of this lecture is to discuss quantum entanglement from the perspective of the theory of Lie groups. More specifically, the ultimate objective of this paper is to quantify quantum entanglement in terms of Lie group invariants, and to make this material accessible to a larger audience than is currently the case. These notes depend extensively on the material presented in Lecture I [28]. It is assumed that the reader is familiar with

the material on density operators and quantum entanglement given in the AMS Short Course Lecture I, i.e., with sections 5 and 7 of [28].

Of necessity, the scope of this paper is eventually restricted to the study of qubit quantum systems, and to a specific problem called the *Restricted Fundamental Problem in Quantum Entanglement* (*RFPQE*). References to the broader scope of quantum entanglement are given toward the end of the paper.

1.1. **Preamble.**
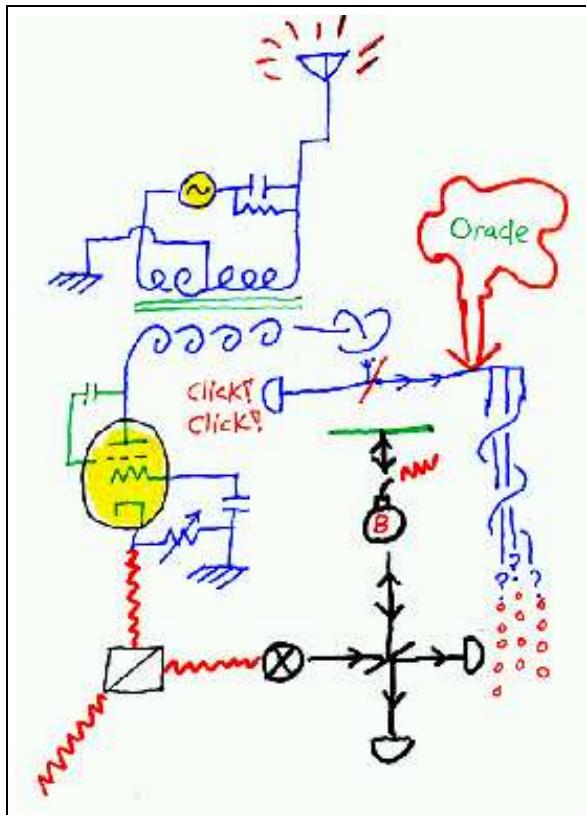
Figure 1. Quantum Entanglement Lab???

At first sight, a physics research lab dedicated to the pursuit of quantum entanglement might look something like the drawing found in Figure 1, i.e., like an indecipherable, incoherent jumble of wires, fiber optic cable, lasers, bean splitters, lenses. Perhaps some large magnets for NMR equipment, or some supercooling equipment for rf SQUIDs are tossed in for good measure. Whatever .. It is indeed a most impressive collection of adult "toys."

However, to a mathematician, such a lab appears very much like a well orchestrated collection of intriguing mathematical "toys," just beckoning with new tantalizing mathematical challenges.

1.2. **A Sneak Preview.**

In the hope of piquing your curiosity to read on, we give the following brief preview of what is to come:

The RFPQE reduces to the mathematical problem of determining the orbits of the big adjoint action of the group of local unitary transformations $\mathbb{L}(2^n)$ on the Lie algebra $u(2^n)$ of the unitary group $\mathbb{U}(2^n)$, as expressed by the following formula:

$$\boxed{\mathbb{L}(2^n) \times u(2^n) \xrightarrow{Ad} u(2^n)}$$

where "Ad" denotes the big adjoint operator, and where the remaining symbols are defined in the table below.

$\mathbb{L}(2^n) = \bigotimes\limits_{1}^{n} \mathbb{SU}(2)$	Local Unitary Group
$\ell(2^n) = \boxplus\limits_{1}^{n} su(2)$	Lie Algebra of $\mathbb{L}(2^n)$
$\mathbb{U}(2^n)$	Unitary Group
$u(2^n)$	Lie Algebra of $\mathbb{U}(2^n)$

We attack this problem by lifting the above big adjoint action to the **induced infinitesimal action**

$$\boxed{\ell(2^n) \xrightarrow{\Omega} \mathbf{Vec}((\mathbf{u}(2^n)))}$$

which, for a 3 qubit density operator ρ, is explicitly given by[1]

$$\boxed{\Omega(v)(i\rho) = \sum_{q_1,q_2=0}^{3} \left(a^{(1)} \cdot x_{*q_1q_2} \times \frac{\partial}{\partial x_{*q_1q_2}} + a^{(2)} \cdot x_{q_1*q_2} \times \frac{\partial}{\partial x_{q_1*q_2}} + a^{(3)} \cdot x_{q_1*q_2} \times \frac{\partial}{\partial x_{q_1q_2*}} \right)}$$

where $v \in \ell(2^3)$ and $i\rho \in \mathbf{u}(2^3)$ are given by

$$\begin{cases} v &= a^{(1)} \cdot \xi_{*00} + a^{(2)} \cdot \xi_{0*0} + a^{(3)} \cdot \xi_{00*} \\ \\ i\rho &= \sum\limits_{r_1,r_2,r_3=0}^{3} x_{r_1r_2r_3} \xi_{r_1r_2r_3} \end{cases}$$

and where $\mathbf{Vec}((\mathbf{u}(2^n)))$ denotes the Lie algebra of vector fields on $\mathbf{u}(2^n)$.

The induced infinitesimal action can then be used to quantify and to classify quantum entanglement through the construction of a complete set of quantum entanglement invariants.

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[1]This expression will be explained later in the paper. I hope that this will make you curious enough to read on?

In the pages to follow, we make every effort to make the above sneak preview more transparent and understandable. Our goal is to present the underlying intuitions without getting lost in an obscure haze of technicalities. However, presenting this topic is much like tiptoeing through a mine field. One false move, and everything explodes into a dense jungle and clutter of technicalities. We leave it to the reader to determine how successful this endeavor is.

1.3. How our view of quantum entanglement has dramatically changed over this past century.

Finally, we close this introduction with a brief historical perspective.

Over the past twentieth century, the scientific community's view of quantum entanglement has dramatically changed. It continues to do so even today.

Initially, quantum entanglement was viewed as an unnecessary and unwanted wart on quantum mechanics. Einstein, Podolsky, and Rosen[13] tried to surgically remove it. Bell[1],[2] showed that such surgery can not be performed without destroying the very life of physical reality.

Today, quantum entanglement is viewed as a useful resource within quantum mechanics. It is now viewed as a commodity to be utilized and traded, much as would be a commodity on the stock exchange.

Quantum entanglement appears to be one of the physical phenomena at the central core of quantum computation. Many believe that it is quantum entanglement that somehow enables us to harness the vast parallelism of quantum superposition.

But what is quantum entanglement?

How do we measure, quantify, classify quantum entanglement? When is the quantum entanglement of two quantum systems the same? different? When is the quantum entanglement of one quantum system greater than that of another?

It is anticipated that answers to the above questions will have a profound impact on the development of quantum computation. Finding answers to these questions is challenging, intriguing, and indeed very habit forming.

2. A Story of Two Qubits, or How Alice & Bob Learn to Live with Quantum Entanglement and Love It.

Our tangled tale of quantum entanglement begins with Alice and Bob's first encounter with quantum entanglement.

Alice and Bob, who happen to be good friends (as attested, time and time again, by the open literature on quantum computation), meet one day. A discussion ensues. The topic, of course, is quantum entanglement. Fortunately or unfortunately, depending on how one looks at it, their discussion explodes into a heated argument. After a lengthy debate, they agree that the only way to resolve their conflict is to purchase the real McCoy, i.e., a pair of entangled qubits. So they rush to the nearest Toys for Aging Children Store to see what they can find.

Almost immediately upon entering the store, they happen to spy, on one of the store shelves, an elaborately decorated box labelled:

$\mathbb{Q}.\mathbb{E}.$, Inc.
Two Entangled Qubits
$\mathcal{Q}_{AB}$
Consisting of qubits
$\mathcal{Q}_A$ and $\mathcal{Q}_B$
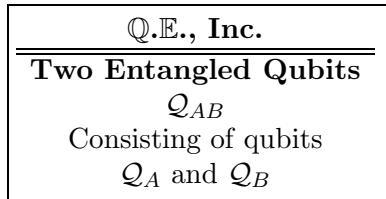
On the back of the box is the content label, required by federal law, which reads:

		U.S. Certified **Contents** EPR Pair $^{(*)}$		
Q Sys	**Hilb. Sp.**	**State**	**Unitary Transf.**	**State Space**
$\mathcal{Q}_{AB}$	$\mathcal{H}_{AB}$	$\rho_{AB} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}$	$\mathbb{U}(2^2)_{AB}$	$u(2^2)_{AB}$
$\mathcal{Q}_A$	$\mathcal{H}_A$	$\rho_A = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	$\mathbb{U}(2)_A$	$u(2)_A$
$\mathcal{Q}_B$	$\mathcal{H}_B$	$\rho_B = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$	$\mathbb{U}(2)_B$	$u(2)_B$
(*) Caveat Emptor: Not legally responsible for the effects of decoherence.				

Alice and Bob hurriedly purchase the two qubit quantum system $\mathcal{Q}_{AB}$. Outside the store, they rip open the box. Alice grabs the qubit labelled $\mathcal{Q}_A$. Bob then takes the remaining qubit $\mathcal{Q}_B$.

Alice and Bob then immediately[2] depart for their separate destinations. Alice flies to Queensland, Australia to continue with her Ph.D. studies at the University of Queensland. She arrives just in time to attend the first class lecture on quantum mechanics. Bob, on the other hand, flies to Vancouver, British Columbia to continue with his Ph.D. studies at the University of British Columbia. He just barely arrives in time to hear the first lecture in a course on differential geometry and Lie groups.

Soon after her quantum mechanics lecture, Alice begins to have second thoughts about their joint purchase of two entangled qubits. She quickly reaches for her cellphone, calls Bob, and nervously fires off in rapid succession three questions:

"Did we get our money's worth of quantum entanglement?"
"How much quantum entanglement did we actually purchase?"
"Are we the victims of a modern day quantum entanglement scam?"

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[2]For some unknown reason, everyone involved with the quantum world is always in a hurry. Perhaps such haste is caused by concerns in regard to decoherence?

After the phone conversation, Bob is indeed deeply concerned. In desperation, he calls the U.S. Quantum Entanglement Protection Agency, which refers him to the U.S. National Institute of Quantum Entanglement Standards and Technology ($NI_{\mathbb{QE}}ST$) in Gaithersburg, Maryland.

After a long conversation, a representative of $NI_{\mathbb{QE}}ST$ agrees to send Alice and Bob, free of charge, the $NI_{\mathbb{QE}}ST$ Quantum Entanglement Standards Kit. On hanging up, the $NI_{\mathbb{QE}}ST$ representative takes the $NI_{\mathbb{QE}}ST$ standard entangled two qubit quantum system $\mathcal{Q}'_{A'B'}$ off the shelf, places $\mathcal{Q}'_{A'}$ together with a User's Manual into a box marked "Alice." He/She also places the remaining qubit $\mathcal{Q}'_{B'}$ together with a User's Manual into a second box labeled "Bob," and then sends the two boxes by overnight mail to Alice and Bob respectively.

The very next day (in different time zones, of course) Alice and Bob each receive their respective packages, take out their respective qubits, and read the enclosed user's manuals.

The $NI_{\mathbb{QE}}ST$ User's Manual reads as follows:

> $\mathbb{Q}.\mathbb{E}$. YARDSTICK 1. An EPR pair $\mathcal{Q}_{AB}$ possess the same quantum entanglement as the $NI_{\mathbb{QE}}ST$ standard EPR pair $\mathcal{Q}'_{A'B'}$ if it is possible for you, Alice and Bob, to use your own local reversible operations (either individually or collectively) to transform $\mathcal{Q}_{AB}$ and $\mathcal{Q}'_{A'B'}$ into one another. If this is possible, then $\mathcal{Q}_{AB}$ and $\mathcal{Q}'_{A'B'}$ are of the **same entanglement type**, written
>
> $$\mathcal{Q}_{AB} \underset{loc}{\sim} \mathcal{Q}'_{A'B'}$$
>
> $\mathbb{Q}.\mathbb{E}$. YARDSTICK 2. An EPR pair $\mathcal{Q}_{AB}$ **possesses more quantum entanglement than** the $NI_{\mathbb{QE}}ST$ standard EPR pair $\mathcal{Q}'_{A'B'}$ if it is possible for you, Alice and Bob, (either individually or collectively) to apply your own reversible and irreversible operations to your respective qubits to transform $\mathcal{Q}_{AB}$ into $\mathcal{Q}'_{A'B'}$. In this case, we write
>
> $$\mathcal{Q}_{AB} \underset{loc}{\geq} \mathcal{Q}'_{A'B'}$$
>
> CAVEAT. Quantum entanglement may be irrevocably lost if Quantum Entanglement Yardstick 2 is applied.

In summary, the above story about Alice and Bob has raised the following questions:

- **Question:** What type of entanglement do Alice and Bob collectively possess?

- **Question:** Is the quantum entanglement of $\mathcal{Q}_{AB}$ the same as the quantum entanglement of $\mathcal{Q}'_{A'B'}$?
- **Question:** Is the quantum entanglement of $\mathcal{Q}_{AB}$ greater than the quantum entanglement of $\mathcal{Q}'_{A'B'}$?

3. LEST WE FORGET, QUANTUM ENTANGLEMENT IS …

Before we continue with our story of Alice and Bob, now is a good opportunity to restate the definition of quantum entanglement found in [28]. Readers not familiar with this definition or related concepts should refer to sections 5 and 7 of [28].

Definition 1. *Let $\mathcal{Q}_1$, $\mathcal{Q}_2$, … , $\mathcal{Q}_n$ be quantum systems with underlying Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, … , $\mathcal{H}_n$, respectively. And let $\mathcal{Q}$ denote the global quantum system consisting of all the quantum systems $\mathcal{Q}_1$, $\mathcal{Q}_2$, … , $\mathcal{Q}_n$, where $\mathcal{H} = \bigotimes_{j=1}^{n} \mathcal{H}_j$ denotes the underlying Hilbert space of $\mathcal{Q}$. Finally let the density operator ρ on the Hilbert space $\mathcal{H}$ denote the state of the global quantum system $\mathcal{Q}$. Then $\mathcal{Q}$ is said to be **entangled** with respect to the Hilbert space decomposition*

$$\mathcal{H} = \bigotimes_{j=1}^{n} \mathcal{H}_j$$

if it can not be written in the form

$$\rho = \sum_{k=1}^{K} \lambda_k \left(\bigotimes_{j=1}^{n} \rho_{(j,k)} \right) \ ,$$

for some positive integer K, where the λ_k's are positive real numbers such that

$$\sum_{k=1}^{K} \lambda_k = 1 \ ,$$

*and where each $\rho_{(j,k)}$ is a density operator on the Hilbert space $\mathcal{H}_j$. If ρ is a pure state, then $\mathcal{Q}$ is **entangled** if ρ can not be written in the form*

$$\rho = \bigotimes_{j=1}^{n} \rho_j \ ,$$

where ρ_j is a density operator on the Hilbert space $\mathcal{H}_j$.

4. Back to Alice and Bob: Local Moves and the Fundamental Problem of Quantum Entanglement (FPQE)

Although the story of Alice and Bob was told with two qubits, the same story could have been told instead with three people, Alice, Bob, Cathy, and three qubits. Or for that matter, it could have equally been told for n people with n qubits. From now on, we will consider the more general story of n people and n qubits.

What Alice, Bob, Cathy, et al were trying to understand can be stated most succinctly as the Fundamental Problem of quantum entanglement, namely:

Fundamental Problem of Quantum Entanglement (**FPQE**). Let ρ and ρ' be density operators representing two different states of a quantum system $\mathcal{Q}$. Is it possible to move $\mathcal{Q}$ from state ρ to state ρ' by applying only **local moves**?

$$\boxed{\text{But what is meant by the phrase ``}\textbf{local move}\text{'' ?}}$$

We define the **standard local moves** as:

Definition 2. *The **standard local moves** are:*

- Local unitary transformations of the form
$$\bigotimes_{k=1}^{n} U_k \in \bigotimes_{k=1}^{n} \mathbb{U}\left(\mathcal{H}_k\right)$$
  For example, for bipartite quantum systems, unitary transformations of the form $U_A \otimes I$, $I \otimes U_B$, $U_A \otimes U_B$
- Measurement of local observables of the form
$$\bigotimes_{k=1}^{n} \mathcal{O}_k \in \bigotimes_{k=1}^{n} Observables\left(\mathcal{H}_k\right)$$

Example 1. *For example, for bipartite quantum systems[3], measurement of local observables of the form $\mathcal{O}_A \otimes I$, $I \otimes \mathcal{O}_B$, $\mathcal{O}_A \otimes \mathcal{O}_B$*

We also define the **extended local moves** as

Definition 3. *The **extended local moves** are:*

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[3]A bipartite quantum system is a global quantum system consisting of two quantum systems.

- Extended local unitary transformations of the form

$$\bigotimes_{k=1}^{n} \mathbb{U}\left(\mathcal{H}_k \otimes \widetilde{\mathcal{H}}_k\right) \ ,$$

where $\mathcal{H}_1, \widetilde{\mathcal{H}}_1, \dots, \mathcal{H}_n, \widetilde{\mathcal{H}}_n$ are distinct non-overhapping Hilbert spaces
- Measurement of extended local observables of the form

$$\bigotimes_{k=1}^{n} Observables\left(\mathcal{H}_k \otimes \widetilde{\mathcal{H}}_k\right) \ ,$$

where $\mathcal{H}_1, \widetilde{\mathcal{H}}_1, \dots, \mathcal{H}_n, \widetilde{\mathcal{H}}_n$ are distinct non-overlapping Hilbert spaces

Definition 4. *Moves based on unitary transformation are called **reversible**. Those based on measurement are called **irreversible**.*

The Horodecki's[17], [18], [19], Jonathan[20], [21], Linden[24], [25], Nielsen[31], [32], [33], [34], Plenio[20], [21], Popescu[24], [25], [48], [27] have made some progress in understanding the FPQE in terms of all four of the above local moves.   For the rest of the talk, we restrict our discussion to reversible standard local moves.

5. A momentary digression: Two different perspectives

Before continuing, it should be mentioned that physics and mathematics approach quantum mechanics from two slightly different but equivalent viewpoints.   To avoid possible confusion, we describe below the minor terminology differences that arise from these two slightly different perspectives.

Physics describes the state of a quantum system in terms of a traceless Hermitian operator ρ, called the density operator.   Observables are Hermitian operators $\mathcal{O}$. Quantum states change via unitary transformations U according to the rubric

$$\rho \longmapsto U\rho U^{\dagger} \ .$$

On the other hand, mathematics describes the state of a quantum system in terms of a skew Hermitian operator $i\rho$, also called the density operator. Observables are skew Hermitian operators $i\mathcal{O}$.   Quantum dynamics are defined via the rule

$$i\rho \longmapsto Ad_U\left(i\rho\right) \ ,$$

where U is a unitary operator lying in the Lie group of unitary transformations $\mathbb{U}\left(N\right)$, and where Ad denotes the big adjoint operator.   Please note that both density operators $i\rho$ and the observables $i\mathcal{O}$ lie in the Lie algebra $u\left(N\right)$ of the unitary group $\mathbb{U}\left(N\right)$.

These minor, but nonetheless annoying differences are summarized in the table below.

Physics		Math
	Hilbert Space $\mathcal{H}$ $Dim(\mathcal{H}) = N$ Unitary Group $\quad \mathbb{U}(N)$ Lie Group	
Observables: $\quad \mathcal{O}$ $\left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right\}$ Density Ops: $\quad \rho$ $N \times N$ Hermitian Ops $A^\dagger = \overline{A}^T = A$	Observables: $\quad i\mathcal{O}$ $\left.\vphantom{\begin{matrix}a\\b\end{matrix}}\right\}$ Density Ops: $\quad i\rho$ $N \times N$ skew Hermitian Ops $\in u(N)$ $(iA)^\dagger = \overline{(iA)}^T = -iA$ where $u(N) = $ Lie algebra of $\mathbb{U}(N)$	
Dynamics via $U \in \mathbb{U}(N)$ $\lvert\psi\rangle \longmapsto U\lvert\psi\rangle$ $\rho \longmapsto U\rho U^\dagger$	Dynamics via $U \in \mathbb{U}(N)$ $\lvert\psi\rangle \longmapsto U\lvert\psi\rangle$ $i\rho \longmapsto Ad_U(i\rho)$ where $Ad_U(i\rho) = U(i\rho)U^{-1}$ is the Big adjoint rep.	

We will use the two different terminologies and conventions interchangeably. Which terminology we are using should be clear from context.

Remark 1. *From* [28] *we know that an element $i\rho$ of the Lie algebra $\mathbf{u}(N)$ is a physical density operator if and only if ρ is positive semi-definite and of trace 1. Thus, the set*

$$\mathbf{density}\,(N) = \{i\rho \in \mathbf{u}(N) \mid \rho \text{ is positive semi-definite of trace 1}\}$$

of physical density operators is a convex subset of the Lie algebra $\mathbf{u}(N)$.

6. The Group of Local Unitary Transformations and the Restricted FPQE

For the sake of clarity of exposition and for the purpose of avoiding minor technicalities, from on we consider only qubit quantum systems, i.e., quantum systems consisting of qubits. The reader, if he/she so wishes, should be able to easily rephrase the results of this paper to more general quantum systems.

Moreover, from this point on, we limit the scope of this talk to the study of quantum entanglement from the perspective of the standard local unitary transformations, i.e., from the perspective of standard reversible local moves

as defined in section 5 of this paper. To emphasize this point, we define the group of local unitary transformations $\mathbb{L}(2^n)$ as follows:

Definition 5. *The **group of local unitary transformations** $\mathbb{L}(2^n)$ is the subgroup of $\mathbb{U}(2^n)$ defined by*

$$\mathbb{L}(2^n) = \bigotimes_1^n \mathbb{SU}(2) \ ,$$

where $\mathbb{SU}(2)$ denotes the special unitary group.

Henceforth, the phrase **"local move"** will mean an element of the group $\mathbb{L}(2^n)$ of **local unitary transformations**..

Convention. From this point on,

$$\boxed{\textbf{Local Moves } = \mathbb{L}(2^n)}$$

Thus, for the rest of this paper we consider only the **Restricted Fundamental Problem of Quantum Entanglement** (**RFPQE**), which is defined as follows:

Restricted Fundamental Problem of Quantum Entanglement (RFPQE). *Let $i\rho$ and $i\rho'$ be density operators lying in the Lie algebra $\mathbf{u}(2^n)$. Does there exist a local move U, i.e., a $U \in \mathbb{L}(2^n)$ such that*

$$i\rho' = U\,(i\rho)\,U^\dagger = Ad_U\,(i\rho) \ ?$$

We will need the following definition:

Definition 6. *Two elements $i\rho$ and $i\rho'$ in $u(2^n)$ are said to be **locally equivalent (or, of the same entanglement type)**, written*

$$i\rho \underset{loc}{\sim} i\rho'$$

provided there exists a $U \in \mathbb{L}(2^n)$ such that

$$i\rho' = Ad_U\,(i\rho) = U\,(i\rho)\,U^{-1}$$

The equivalence class

$$[i\rho]_E = \left\{ i\rho' \mid i\rho \underset{loc}{\sim} i\rho' \right\}$$

*is called an **entanglement class** (or, an **orbit** of the big adjoint action of $\mathbb{L}(2^n)$ on the Lie algebra $\mathbf{u}\,(2^n)$). Finally, let*

$$\mathbf{u}(2^n)/\mathbb{L}(2^n)$$

*denote the **set of entanglement classes**.*
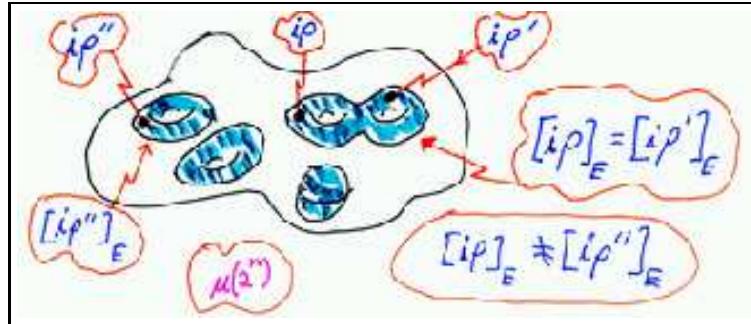
Figure 2. Quantum entanglement classes.

The entanglement classes of the Lie algebra $\mathbf{u}(2^n)$ are just the **orbits** of the big adjoint action of $\mathbb{L}(2^n)$ on $\mathbf{u}(2^n)$. Two states are entangled in the same way if and only if they lie in the same entanglement class, i.e., in the same orbit.

Remark 2. *Local unitary transformations can not entangle quantum systems with respect to the above tensor product decomposition. However, **global unitary transformations** (i.e., unitary transformations lying in $\mathbb{U}(2^n) - \mathbb{L}(2^n)$) are those unitary transformations which can and often do produce interactions which entangle quantum systems.*

$$\boxed{\textbf{But what is quantum entanglement?}}$$

7. Summary and List of Objectives

We are now in a position to state clearly the main objectives of this paper. Namely, in regard to the **Restricted Fundamental Problem of Quantum Entanglement (RFPQE)**, our objectives are twofold:

Objective 1. Given a density operator $i\rho$, devise a means of determining the dimension of its entanglement class $[i\rho]_E$. We will accomplish this by determining the dimension of the tangent plane $T_{i\rho}[i\rho]_E$ to the manifold $[i\rho]_E$ at the point $i\rho$.

Objective 2. Given two states $i\rho$ and $i\rho'$, devise a means of determining whether they belong to the same or different entanglement class. We will accomplish this by constructing a **complete set of quantum entanglement invariants**, i.e., invariants that completely specify all the orbits (i.e., all the entanglement classes). In this sense, we have completely quantified quantum entanglement. In other words, find a finite

set $\{f_1, f_2, \ldots, f_K\}$ of real valued functions on $\mathbf{u}(2^n)$ which distinguish all entanglement classes, i.e.,

$$i\rho \underset{loc}{\sim} i\rho' \iff f_k(i\rho) = f_k(i\rho') \text{ for every } k.$$

8. If you are unfamiliar with … , then make a quantum jump to Appendices A & B

This section is meant to play the role of a litmus test for the reader. If the reader feels reasonably comfortable with the concepts listed below, then it is suggested that the reader proceed to the next section of this paper. If not, it is strongly suggested that the reader read Appendices A and B of this paper before proceeding to the next section.

Let $\mathbb{G}$ be a Lie group, and let $\mathfrak{g}$ denote its Lie algebra.

8.1. **Litmus Test 1. The exponential map.**

The reader should be familiar with the exponential map

$$\exp : \mathfrak{g} \longrightarrow \mathbb{G} \ ,$$

which for matrix Lie Groups is given by the power series

$$\exp(M) = \sum_{k=0}^{\infty} \frac{1}{k!} M^k$$

8.2. **Litmus Test 2. The Lie bracket.**

The reader should be familiar with the Lie bracket

$$[-,-] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g} \ ,$$

which for matrix Lie groups is given by the commutator

$$[A, B] = AB - BA$$

8.3. **Litmus Test 3. The Lie algebra under three different guises.**

The Lie algebra $\mathfrak{g}$ of the Lie group $\mathbb{G}$ can be viewed in each of the following mathematically equivalent ways:

- As $T_I\mathbb{G}$, i.e., as the tangent space to the Lie group $\mathbb{G}$ at the identity I.
- As $\mathbf{Vec}_R(\mathbb{G})$, i.e., as the Lie algebra of right invariant smooth vector fields on the Lie group $\mathbb{G}$.
- As $\mathbf{Der}_\infty(\mathbb{G})$, i.e., as the Lie algebra of all derivations (i.e., directional derivatives) on the algebra $C^\infty(\mathbb{G})$ of all smooth real valued functions on $\mathbb{G}$.

In summary,

$$\boxed{\mathfrak{g} = T_I\mathbb{G} = \mathbf{Vec}_R(\mathbb{G}) = \mathbf{Der}_\infty(\mathbb{G})}$$

If you feel comfortable with the above three litmus tests, then please proceed to the next section.

9. DEFINITION OF QUANTUM ENTANGLEMENT INVARIANTS

Let $C^\infty(\mathbf{u}(2^n))$ denote the algebra of smooth (C^∞) real valued functions on the Lie algebra $\mathbf{u}(2^n)$, i.e.,

$$C^\infty(\mathbf{u}(2^n)) = \{f : \mathbf{u}(2^n) \longrightarrow \mathbb{R} \mid f \text{ is smooth}\}$$

Definition 7. *A function $f \in C^\infty(\mathbf{u}(2^n))$ is called a **(quantum) entanglement invariant** if f is invariant under the big adjoint action of $\mathbb{L}(2^n)$, i.e., if*

$$f(Ad_U(i\rho)) = f(i\rho)$$

for all $U \in \mathbb{L}(2^n)$, and for all $i\rho$ in $\mathbf{u}(2^n)$. The collection of all (quantum) entanglement invariants forms an algebra, which we denote by

$$C^\infty(\mathbf{u}(2^n))^{\mathbb{L}(2^n)} \ .$$

Definition 8. *A subset $\{f_1, f_2, \ldots, f_m\}$ of $C^\infty(\mathbf{u}(2^n))^{\mathbb{L}(2^n)}$ is called a **complete set of entanglement invariants** if*

$$i\rho \underset{loc}{\sim} i\rho' \text{ iff } f_k(i\rho) = f_k(i\rho') \text{ for all } f_k \text{ in } \{f_1, f_2, \ldots, f_m\} \ .$$

Definition 9. *Let $\mathcal{P}(\mathbf{u}(2^n))$ be the subalgebra of $C^\infty(\mathbf{u}(2^n))$ of all functions $f \in C^\infty(\mathbf{u}(2^n))$ which are polynomial functions, i.e., of all functions f for which $f(v)$ is a polynomial function of the entries in v. We define the **algebra of polynomial entanglement invariants** as*

$$\mathcal{P}(\mathbf{u}(2^n))^{\mathbb{L}(2^n)} = \mathcal{P}(\mathbf{u}(2^n)) \cap C^\infty(\mathbf{u}(2^n))^{\mathbb{L}(2^n)}$$

Theorem 1. $\mathcal{P}\left(\mathbf{u}(2^n)\right)^{\mathbb{L}(2^n)}$ *is a finitely generated algebra.*

Definition 10. *A minimal set of generators of* $\mathcal{P}\left(\mathbf{u}(2^n)\right)^{\mathbb{L}(2^n)}$ *is called a* **basic set of entanglement invariants**.

Remark 3. *It is important to note that a basic set of entanglement invariants is not always a complete set of entanglement invariants. We will show that this is the case for entanglement invariants for two qubit quantum systems.*

10. THE LIE ALGEBRA $\ell\left(2^n\right)$ OF $\mathbb{L}(2^n)$

To understand and work with the big adjoint action

$$\mathbb{L}\left(2^n\right) \times \mathbf{u}\left(2^n\right) \xrightarrow{Ad} \mathbf{u}\left(2^n\right)$$

we will need to lift this action to the corresponding infinitesimal action of the Lie algebra $\ell\left(2^n\right)$ of $\mathbb{L}(n)$. The Lie algebra $\ell\left(2^n\right)$ will play a crucial role in our achieving objectives 1 and 2 as stated in the previous section.

Definition 11. *The Lie algebra* $\ell\left(2^n\right)$ *is the (real) Lie algebra given by the following Kronecker sum*

$$\ell\left(2^n\right) = \underbrace{\mathbf{su}\left(2\right) \boxplus \mathbf{su}\left(2\right) \boxplus \cdots \boxplus \mathbf{su}\left(2\right)}_{n\ terms} \ ,$$

where $\mathbf{su}\left(2\right)$ *denotes the Lie algebra of the special unitary group* $\mathbb{SU}\left(2\right)$, *and where the* **Kronecker sum** '$A \boxplus B$' *of two matrices (or operators)* A *and* B *is defined by*

$$A \boxplus B = A \otimes \mathbf{1} + \mathbf{1} \otimes B \ ,$$

with '$\mathbf{1}$' *denoting the identity matrix (or operator).*

A basis[4] of the (real) Lie algebra $\mathbf{u}\left(2^n\right)$ is given by

$$\{\xi_{k_1 k_2 \ldots k_n} \mid k_1, k_2, \ldots, k_n = 0, 1, 2, 3\} \ ,$$

where

$$\xi_{k_1 k_2 \ldots k_n} = -\frac{i}{2}\sigma_{k_1} \otimes \sigma_{k_2} \otimes \cdots \otimes \sigma_{k_n} \ ,$$

and where

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

---

[4]For more information, please refer to Appendix B.

denote the Pauli spin matrices, and where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

denotes the 2×2 identity matrix.

It follows that a basis of the Lie algebra $\ell\left(2^n\right)$ as a subalgebra of $\mathbf{u}\left(2^n\right)$ is:

$$\{\xi_{k_1 k_2 \ldots k_n} \mid k_1, k_2, \ldots, k_n = 0, 1, 2, 3, \text{ where exactly one } k_j \neq 0\}$$

For example,

- $\{\xi_1, \xi_2, \xi_3\}$ is a basis of $\ell\left(1\right)$
- $\{\xi_{01}, \xi_{02}, \xi_{03}, \xi_{10}, \xi_{20}, \xi_{30}\}$ is a basis of $\ell\left(2\right)$
- $\{\xi_{001}, \xi_{002}, \xi_{003}, \xi_{010}, \xi_{020}, \xi_{030}, \xi_{100}, \xi_{200}, \xi_{300}\}$ is a basis of $\ell\left(3\right)$

Thus, we have the following proposition:

Proposition 1. *The Lie algebra $\ell\left(2^n\right)$ of the Lie group $\mathbb{L}\left(2^n\right)$ of local unitary transformations is of dimension $3n$, i.e.,*

$$Dim\left(\ \ell\left(2^n\right)\ \right) = 3n$$

11. Definition of the Infinitesimal Action

We now show how quantum entanglement invariants can be found by lifting the big adjoint action to the Lie algebra $\ell\left(2^n\right)$, where the problem becomes a linear one.

The big adjoint action

$$\mathbb{L}\left(2^n\right) \times \mathbf{u}\left(2^n\right) \xrightarrow{Ad} \mathbf{u}\left(2^n\right)$$

induces an **infinitesimal action**

$$\ell\left(2^n\right) \xrightarrow{\Omega} \mathbf{Vec}\left(\mathbf{u}(2^n)\right)$$

as follows.

Let $v \in \ell\left(2^n\right)$. We define the vector field $\Omega\left(v\right)$ on $\mathbf{u}\left(2^n\right)$ by constructing a tangent vector $\Omega\left(v\right)|_{i\rho}$ for each $i\rho \in \mathbf{u}\left(2^n\right)$.

Let $\gamma_v\left(t\right)$ be the smooth curve in $\mathbf{u}\left(2^n\right)$ defined by

$$\gamma_v\left(t\right) = Ad_{\exp(tv)}\left(i\rho\right)\ .$$

Then $\gamma_v\left(t\right)$ is a curve which passes through $i\rho$ at time $t = 0$. We define $\Omega\left(v\right)|_{i\rho}$ as the tangent vector to $\gamma_v\left(t\right)$ at $t = 0$.
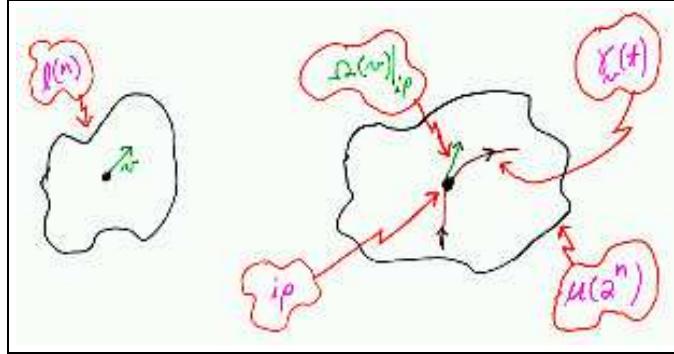
Figure 3.   Induced infinitesimal action.

But what is the meaning of the infinitesimal action

$$\ell\left(2^n\right) \xrightarrow{\ \Omega\ } \mathbf{Vec}\left(\mathbf{u}(2^n)\right)$$

that we have just defined?

Each $\Omega\left(i\rho\right)\big|_{i\rho}$ is a direction in $\mathbf{u}\left(2^n\right)$ from $i\rho$ that we can move without leaving the quantum entanglement class $[i\rho]_E$. Movement in all directions not in $\mathrm{Im}\left(\Omega\right)\big|_{i\rho}$ will force us to immediately leave $[i\rho]_E$.
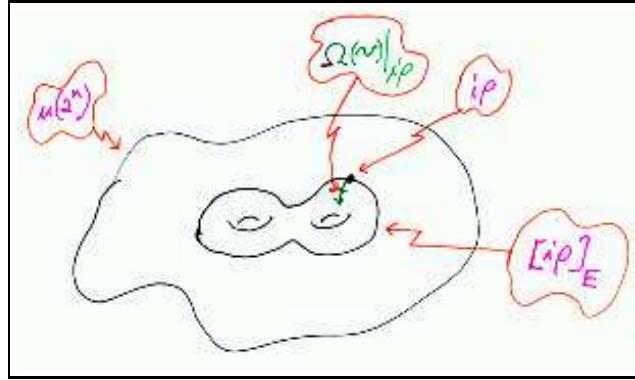
Figure 4.   Moving in a direction that stays within the entanglement class.

As the reader might expect, the infinitesimal action Ω can naturally be expressed in terms of the small adjoint operator ad. In particular, we have:

Proposition 2. $\Omega\left(v\right) = ad_v$ *for all v in the Lie algebra $\ell\left(2^n\right)$*

Proof. Let v be an arbitrary element of the Lie algebra $\ell\left(2^n\right)$, and let $i\rho$ be an arbitrary element of the Lie algebra $\mathbf{u}\left(2^n\right)$.

By definition, $\Omega\left(v\right)\left(i\rho\right)$ is the tangent vector at $t=0$ to the curve $\gamma_v\left(t\right)$ in $\mathbf{u}\left(2^n\right)$ given by

$$\gamma_v\left(t\right) = Ad_{\exp(tv)}\left(i\rho\right)\ .$$

Hence,

$$
\begin{aligned}
\Omega\left(v\right)\left(i\rho\right) &= \left.\tfrac{d}{dt}\, Ad_{\exp(tv)}\left(i\rho\right)\right|_{t=0} \\[2ex]
&= \left.\tfrac{d}{dt}\, \exp\left(ad_{tv}\right)\left(i\rho\right)\right|_{t=0} \\[2ex]
&= \left.\tfrac{d}{dt}\, \exp\left(t \cdot ad_{v}\right)\left(i\rho\right)\right|_{t=0} \\[2ex]
&= \left.\tfrac{d}{dt}\, \left(1 + t \cdot ad_{v} + o(t^2)\right)\left(i\rho\right)\right|_{t=0} \\[2ex]
&= ad_{v}\left(i\rho\right)
\end{aligned}
$$

$\square$

The above formula will prove to be useful when we actually calculate the entanglement invariants of some examples given in later sections.

12. The significance of the infinitesimal action Ω

As stated in the appendices, the Lie algebra $\mathbf{Vec}\left(\mathbf{u}\left(2^n\right)\right)$ of all smooth vector fields on $\mathbf{u}\left(2^n\right)$ can be identified with the Lie algebra of derivations $\mathbf{Der}\left(C^\infty \mathbf{u}\left(2^n\right)\right)$.

The significance of the infinitesimal action

$$
\Omega : \ell\left(2^n\right) \longrightarrow \mathbf{Vec}\left(\mathbf{u}\left(2^n\right)\right)
$$

is best expressed in terms of the following theorem:

Theorem 2. *Let*

$$
\{v_1, v_2, \ldots, v_{3n}\}
$$

be a basis for the Lie algebra $\ell\left(2^n\right)$. Then a smooth real valued function

$$
f : \mathbf{u}\left(2^n\right) \longrightarrow \mathbb{R}
$$

is an entanglement invariant if and only if it satisfies the following system of partial differential equations

$$
\begin{cases}
\Omega\left(v_1\right)f &= 0 \\[2ex]
\Omega\left(v_2\right)f &= 0 \\[2ex]
\quad \vdots \qquad \vdots \;\; \vdots \\[2ex]
\Omega\left(v_{3n}\right)f &= 0
\end{cases}
$$

The intuition underlying the above theorem is that $\Omega\left(v_1\right)$, $\Omega\left(v_2\right)$, ... , $\Omega\left(v_{3n}\right)$ are the linearly independent directions we can move without leaving the entanglement class we are currently in. Hence, if f is an entanglement invariant, then its rate of change (i.e., its directional derivative) in each of the directions $\Omega\left(v_1\right)$, $\Omega\left(v_2\right)$, ... , $\Omega\left(v_{3n}\right)$ must be zero, and vice versa.

This theorem provides us with a means of determining a complete set of entanglement invariants. All that we need to do is to solve the above system of partial differential equations.

13. Achieving our two objectives, ... finally

We now show how the infinitesimal action

$$\ell\left(2^n\right) \xrightarrow{\Omega} \mathbf{Vec}\left(\mathbf{u}(2^n)\right)$$

can be used to achieve the two objectives listed in section 8 of this paper.

Objective 1. Given an arbitrary density operator $i\rho$, devise a means of determining the dimension of its entanglement class $[i\rho]_E$.

Objective 1 is achieved as follows:

We begin by noting that $\mathbf{Vec}\left(\mathbf{u}\left(2^n\right)\right)|_{i\rho}$ is the same as the tangent space $T_{i\rho}\left(\mathbf{u}(2^n)\right)$ to $\mathbf{u}\left(2^n\right)$ at the point $i\rho$, and that $\mathrm{Im}\left(\Omega\right)|_{i\rho}$ is the same as the tangent space $T_{i\rho}\left([i\rho]_E\right)$ to the entanglement class $[i\rho]_E$ at $i\rho$. Hence, the dimension of $[i\rho]_E$ is same as the dimension as its tangent space at $i\rho$, i.e.,

$$Dim\left([i\rho]_E\right) = Dim\left(T_{i\rho}\left([i\rho]_E\right)\right) = Dim\left(\mathrm{Im}\left(\Omega\right)|_{i\rho}\right)$$

The task of finding the dimension of the entanglement class $[i\rho]_E$ reduces to that of computing the dimension of the vector space $\mathrm{Im}\left(\Omega\right)|_{i\rho}$. We will give examples of this dimension calculation in the next two sections.

We next use the infinitesimal action to achieve:

Objective 2. Given two states $i\rho$ and $i\rho'$, devise a means of determining whether they belong to the same or different entanglement class.

as follows:

We begin by noting that $\mathbf{Vec}\left(\mathbf{u}\left(2^n\right)\right)$ can be identified with the Lie algebra $Der\left(C^\infty\mathbf{u}(2^n)\right)$ of derivations on $\mathbf{u}\left(2^n\right)$. Next we recall that $\mathrm{Im}\left(\Omega\right)$ consists of all directions in $\mathbf{u}\left(2^n\right)$ that we can move without leaving an entanglement class that we are in. If

$$f \in \left(C^\infty\left(\mathbf{u}(2^n)\right)\right)^{\mathbb{L}(2^n)}$$

is an entanglement invariant, then f will not change if we move in any direction within $\text{Im}(\Omega)$. As a result we have the following theorem:

Theorem 3. *Let $v_1, v_2, \ldots, v_{3n}$ be a vector space basis of the (real) Lie algebra $\ell(2^n)$. Then*

$$f \in (C^\infty(\mathbf{u}(2^n)))^{\mathbb{L}(2^n)} \Longleftrightarrow \Omega(v_j) f = 0$$

for all j, where $\Omega(v_j)$ is interpreted as a differential operator in $Der(C^\infty \mathbf{u}(2^n))$.

In other words, the task of finding entanglement invariants reduces to that of solving a system of linear partial differential equations. We will give examples of this calculation in the examples found in the next two sections of this paper.

14. EXAMPLE 1. THE ENTANGLEMENT CLASSES OF $n = 1$ QUBITS

We now make use of the methods developed in the previous section to study the entanglement classes associated with $n = 1$ qubits. This is a trivial but nonetheless instructive case. As we shall see, there is no entanglement in this case. But there are many entanglement classes!

For this example, the local unitary group $\mathbb{L}(2^1)$ is the same as the special unitary group $\mathbb{SU}(2^1)$. The corresponding Lie algebra $\ell(2^1)$ is the same as the Lie algebra $\mathbf{su}(2)$. Each density operator $i\rho$ lies in the Lie algebra $\mathbf{u}(2^1)$.

As an immediate consequence of Proposition 2 of Section 11, the infinitesimal action

$$\Omega : \ell(2^1) \longrightarrow \mathbf{Vec}(\mathbf{u}(2))$$

is simply the small adjoint action, i.e.,

$$\Omega(v) = ad_v \ ,$$

for all $v \in \ell(2^1)$.

We can now use the bases[5]

$$\left\{ \xi_1 = -\tfrac{1}{2}\sigma_1, \ \xi_2 = -\tfrac{1}{2}\sigma_2, \ \xi_3 = -\tfrac{1}{2}\sigma_3 \right\}$$
$$\text{and}$$
$$\left\{ \xi_0 = -\tfrac{1}{2}\sigma_0, \ \xi_1 = -\tfrac{1}{2}\sigma_1, \ \xi_2 = -\tfrac{1}{2}\sigma_2, \ \xi_3 = -\tfrac{1}{2}\sigma_3 \right\}$$

of the respective Lie algebras $\ell(2^1)$ and $\mathbf{u}(2)$ to find a more useful expression for $\Omega(v)$.

---

[5]See Section 10.

Each element $v \in \ell\left(2^1\right)$ can be uniquely expressed in the form

$$v = a \cdot \xi \ ,$$

where $a = (a_1, a_2, a_3) \in \mathbb{R}^3$ and $\xi = (\xi_1, \xi_2, \xi_3)$. Thus,

$$\Omega\left(v\right) = \Omega\left(a \cdot \xi\right) = ad_{a \cdot \xi} = a \cdot ad_\xi \ ,$$

where

$$ad_\xi = \left(ad_{\xi_1}, ad_{\xi_2}, ad_{\xi_3}\right) \ .$$

Moreover, each element $i\rho \in \mathbf{u}\left(2\right)$ can be uniquely written in terms of the basis of $\mathbf{u}\left(2\right)$ as

$$i\rho = x_0 \xi_0 + x \cdot \xi \ ,$$

where $x = (x_1, x_2, x_3)$ and $\xi = (\xi_1, \xi_2, \xi_3)$.

In terms of the basis of $\mathbf{u}\left(2\right)$,

$$ad_{\xi_j} = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & L_j \end{pmatrix} = 0 \oplus L_j & \text{if } j = 1, 2, 3 \\[2ex] \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 & \text{if } j = 0 \end{cases} \ ,$$

where

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the basis[6] of the Lie algebra $\mathbf{so}\left(3\right)$ of the special orthogonal group $\mathbb{SO}\left(3\right)$ given in Appendix B.

Let

$$\left\{\frac{\partial}{\partial x_0}, \ \frac{\partial}{\partial x_1}, \ \frac{\partial}{\partial x_2}, \ \frac{\partial}{\partial x_3}\right\}$$

denote the basis[7] of $\mathbf{Vec}\left(\mathbf{u}\left(2\right)\right)$ induced by the chart

$$\mathbf{u}\left(2\right) \qquad \xrightarrow{\ \pi\ } \qquad \mathbb{R}^4$$

$$i\rho = \textstyle\sum_{j=0}^3 x_j \xi_j \quad \longmapsto \quad (x_0, x_1, x_2, x_3) = (x_0, x) \ .$$

In other words, for each j, $\partial/\partial x_j$ denotes the vector field on $\mathbf{u}\left(2\right)$ defined at each point $i\rho$ as the tangent vector to the curve $\pi^{-1}\left(x_0, \ldots, x_j + t, \ldots, x_3\right) = i\rho + t\xi_j$ at $t = 0$.

---

[6]This follows from the following calculation:

$$\begin{aligned} ad_{\xi_j}\left(\xi_k\right) & = & ad_{-i\sigma_j/2}\left(-i\sigma_k/2\right) & = & \left[-i\sigma_j/2, -i\sigma_k/2\right] & = & -\tfrac{1}{4}\left[\sigma_j, \sigma_k\right] \\ & = & -\tfrac{1}{2} i \epsilon_{jkp} \sigma_p & = & \epsilon_{jkp} \xi_p \end{aligned}$$

where $L_j = \left(\epsilon_{jkp}\right)$.

[7]For those unfamiliar with this basis, please refer to Appendix A page 37.

Then,

$$\Omega\left(v\right)\left(i\rho\right) = (x_0, x) \cdot (0 \oplus a \cdot L) \cdot \begin{pmatrix} \partial/\partial x_0 \\ \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix},$$

$$= x \cdot (a \cdot L) \cdot \triangledown$$

$$= a \cdot x \times \triangledown,$$

where '$\times$' denotes the vector cross product, and where

$$L = (L_1, L_2, L_3) \text{ and } \triangledown = \begin{pmatrix} \partial/\partial x_1 \\ \partial/\partial x_2 \\ \partial/\partial x_3 \end{pmatrix}.$$

We can now achieve objective 1.

Objective 1.   *Given an arbitrary density operator $i\rho$ in $\mathbf{u}(2)$, find the dimension of an arbitrary entanglement class $[i\rho]_E$.*

From the above discussion, it follows that the image $\text{Im}(\Omega)$ of the infinitesimal action Ω is spanned by the three vector fields

$$\begin{cases} \Omega\left(\xi_1\right) &= x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} \\[2mm] \Omega\left(\xi_2\right) &= x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1} \\[2mm] \Omega\left(\xi_3\right) &= x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} \end{cases},$$

defined on $\mathbf{u}(2^n) = \left\{ i\rho = \sum_{j=0}^{3} x_j \xi \mid x_0, x_1, x_2, x_3 \in \mathbb{R} \right\}$. In particular, the tangent space $T_{i\rho}\left([i\rho]_E\right) = (\text{Im}\,\Omega)|_{i\rho}$ of the entanglement class $[i\rho]_E$ at the point $i\rho$ is spanned by

$$\Omega\left(\xi_1\right)|_{i\rho}, \ \Omega\left(\xi_2\right)|_{i\rho}, \ \Omega\left(\xi_3\right)|_{i\rho}$$

As can be easily verified by the reader, the above three vectors span a two dimensional space if $|x| \neq 0$ and a zero dimensional vector space if $|x| = 0$.

Since $(\text{Im}\,\Omega)|_{i\rho}$ is the tangent space $T_{i\rho}\left([i\rho]_E\right)$ of $[i\rho]_E$ at the point $i\rho$, and since the dimension of $[i\rho]_E$ is the same as the dimension of its tangent space $T_{i\rho}\left([i\rho]_E\right)$ at $i\rho$, it follows that the dimension of the entanglement class $[i\rho]_E$ is given by:

$$Dim\,[i\rho]_E = \begin{cases} 2 & \text{if } |x| \neq 0 \\[2mm] 0 & \text{if } |x| \neq 0 \end{cases}$$

We are now ready to achieve objective 2:

Objective 2. *Given two states $i\rho$ and $i\rho'$, devise a means of determining whether they belong to the same or different entanglement class.*

We achieve this objective by determining a complete set of entanglement invariants[8] for one qubit quantum systems, i.e., by determining a set of entanglement invariants $\{f_1, f_2, \ldots, f_k\}$ such that

$$i\rho \underset{loc}{\sim} i\rho' \text{ if and only if } f_j(i\rho) = f_j(i\rho') \text{ for all } j .$$

.

We begin by recalling that the Lie algebra $\mathbf{Vec}(\mathbf{u}(2))$ of vector fields on $\mathbf{u}(2)$ can be identified with the Lie algebra $\mathbf{Der}(C^\infty \mathbf{u}(2))$ of all derivations on the smooth real valued functions on $\mathbf{u}(2)$. Thus, the elements of $\operatorname{Im}\Omega$ can be viewed as directional derivatives, directional derivatives in those directions in which we can move and still remain in the same entanglement class.

From theorem 2, it immediately follows that a real valued function $f : \mathbf{u}(2) \longrightarrow \mathbb{R}$ is an entanglement invariant if and only it is a solution of the system of partial differential equations (PDEs):

$$\begin{cases} \Omega(\xi_1) f = 0 \\[2mm] \Omega(\xi_2) f = 0 \\[2mm] \Omega(\xi_3) f = 0 \end{cases}$$

Since from above we know that $\Omega(\xi_j)(i\rho) = x \cdot L_j \cdot \triangledown$, we can write the above system of PDEs more explicitly as:

$$\begin{cases} x_3 \frac{\partial f}{\partial x_2} - x_2 \frac{\partial f}{\partial x_3} = 0 \\[2mm] x_1 \frac{\partial f}{\partial x_3} - x_3 \frac{\partial f}{\partial x_1} = 0 \\[2mm] x_2 \frac{\partial f}{\partial x_1} - x_1 \frac{\partial f}{\partial x_2} = 0 \end{cases} ,$$

where, as before, $i\rho = x_0 \xi_0 + x \cdot \xi$.

From theorem 2, we know that a complete set of quantum entanglement invariants for one qubit systems is the same as a complete functionally independent set of solutions of the above system of PDEs. Thus, solving the above system of PDEs by standard methods found in the theory of differential equations, we find that

$$\left\{ f(x) = \sqrt{x_1^2 + x_2^2 + x_3^2} \right\}$$

is a complete set of entanglement invariants.

---

[8] As we shall see, in this particular case of $n = 1$ qubits, the complete set of entanglement invariants consists of only one invariant.

A functionally equivalent complete set of entanglement invariants is

$$\left\{ f' = x_1^2 + x_2^2 + x_3^2 \right\} \ ,$$

which is also a basic set of entanglement invariants.

Remark 4. *Fortunately, in this simplest case, a complete set of entanglement invariants and a basic set of entanglement invariants are one and the same. This will not be the case for quantum systems of more than one qubit.*

14.1. **The Bloch "sphere".** As a result of the previous calculation, we have a complete set of entanglement invariants, namely

$$f(x) = \sqrt{x_1^2 + x_2^2 + x_3^2} = |x|$$

We have completely classified all the entanglement classes for 1 qubit quantum systems. For in this case,

$$[i\rho]_E = \left[i\rho' \right] \Longleftrightarrow f(i\rho) = f(i\rho') \ .$$

As a consequence of this result, the induced foliation of the space **density** $\left(2^1 \right)$ of all physical density operators lying in in the Lie algebra $\mathbf{u}\left(2^1 \right)$ can be visualized in terms of the 3-ball or radius 1 in $\mathbb{R}^3$, called the **Bloch "sphere."**

Recall from remark on page 12 that

$$\mathbf{density}\left(2^1 \right) = \left\{ i\rho \in \mathbf{u}\left(2^1 \right) \mid \rho \text{ is positive semi-definite and of trace one} \right\}$$

is a convex subset of the of the Lie algebra $\mathbf{u}\left(2^1 \right)$. In this special case of $n = 1$ qubit, it is a straight forward exercise to show that

$$\mathbf{density}\left(2^1 \right) = \left\{ i\rho = x_0\xi_0 + x \cdot \xi \mid |x| \leq 1 \text{ and } x_0 = -1 \right\} \ .$$

Thus, the convex subset **density** $\left(2^1 \right)$ of $\mathbf{u}\left(2^1 \right)$ of all physical density operators $i\rho$ in $\mathbf{u}\left(2^1 \right)$ can naturally be identified with the 3-ball of radius one via the one-to-one correspondence

$$i\rho = x_0\xi_0 + x_1\xi_1 + x_2\xi_2 + x_3\xi_3 \longleftrightarrow (x_1, x_2, x_3)$$

as illustrated in Figure 5.
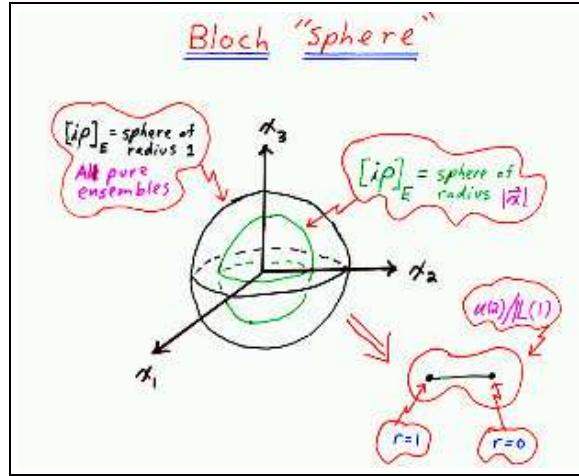
Figure 5. The Bloch "sphere"

It follows that each entanglement class $[i\rho]_E$ is simply a sphere of radius $f(i\rho) = |x|$.   The sphere of radius one is the entanglement class of all pure ensembles.   All other spheres represent entanglement classes of mixed ensembles. The "sphere" of radius 0 (i.e., the origin) represents the entanglement class of the maximally mixed ensemble.

IAnd so we can conclude that the space of entanglement classes lying in the space formed by identifying the elements of the convex set **density** $\left(2^1\right)$ via the action of the local transformation group $\mathbb{L}\left(2^1\right)$, namely

$$\textbf{density}\left(2^1\right)/\mathbb{L}\left(2^1\right) \ ,$$

is simply a closed[9] line segment.

In terms of this picture, it is easy to visualize the tangent space $T_{i\rho}\left([i\rho]_E\right)$ to $[i\rho]_E$ at $i\rho$. Moreover, it is easy to visualize the normal bundle of $[i\rho]_E$. For the normal vector field is simply

$$x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}\bigg|_{[i\rho]_E}$$

Unfortunately, for quantum systems of more than one qubit, such a visualization is by no means as easy.

---

[9]The adjective "closed" means that the line segment contains both its endpoints.

15. EXAMPLE 2. THE ENTANGLEMENT CLASSES OF $n = 2$ QUBITS

As one might expect, the entanglement of two qubit quantum systems is much more complex than that of one qubit quantum systems. In fact, with each additional qubit, the entanglement becomes exponentially more complex than before. Perhaps this is a strong hint as to where the power of quantum computation is coming from?

For this example, the local unitary group $\mathbb{L}\left(2^2\right)$ is the Lie group $\mathbb{SU}\left(2^1\right) \otimes \mathbb{SU}\left(2^1\right)$. The corresponding Lie algebra $\ell\left(2^2\right)$ is Kronecker sum[10] $\mathbf{su}(2) \boxplus \mathbf{su}(2)$. Each density operator $i\rho$ lies in the Lie algebra $\mathbf{u}\left(2^2\right)$.

As an immediate consequence of Proposition 2 given in Section 11, the infinitesimal action

$$\Omega : \ell\left(2^2\right) \longrightarrow \mathbf{Vec}\left(\mathbf{u}\left(2^2\right)\right)$$

is simply the small adjoint action, i.e.,

$$\Omega\left(v\right) = ad_v \ ,$$

for all $v \in \ell\left(2^2\right)$.

We can now use the bases[11]

$$\{\xi_{10}, \xi_{20}, \xi_{30}, \xi_{01}, \xi_{02}, \xi_{03}, \}$$
$$\text{and}$$
$$\{\xi_{ij} \mid i, j = 0, 1, 2, 3\}$$

of the respective Lie algebras $\ell\left(2^2\right)$ and $\mathbf{u}\left(2^2\right)$ to find a more useful expression for $\Omega\left(v\right)$, where

$$\xi_{ij} = -\frac{i}{2}\sigma_i \otimes \sigma_j \ .$$

.

Each element $v \in \ell\left(2^2\right)$ can be uniquely expressed in the form

$$v = (a \cdot \xi) \otimes I_4 + I_4 \otimes (b \cdot \xi) = a \cdot \xi \boxplus b \cdot \xi \ ,$$

---

[10]We remind the reader that the Kronecker sum $A \boxplus B$ of two matrices (operators) A and B is defined as

$$A \boxplus B = A \otimes \mathbf{1} + \mathbf{1} \otimes B$$

where $\mathbf{1}$ denotes the identity matrix (operator).

[11]See Section 10.

where $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3,)$ lie in $\mathbb{R}^3$, where $\xi = (\xi_1, \xi_2, \xi_3)$, and where I_4 is the 4×4 identity matrix. Thus,

$$
\begin{aligned}
\Omega(v) &= \Omega\left(\sum_{j=1}^{3}(a_j\xi_{j0} + b_j\xi_{0j})\right) \\[2mm]
&= \Omega(a \cdot \xi \boxplus b \cdot \xi) \\[2mm]
&= ad_{a \cdot \xi \boxplus b \cdot \xi} \\[2mm]
&= ad_{(a \cdot \xi) \otimes I_4} + ad_{I_4 \otimes (b \cdot \xi)} \\[2mm]
&= I_4 \otimes (a \cdot ad_\xi) + (b \cdot ad_\xi) \otimes I_4
\end{aligned}
$$

where

$$
ad_\xi = (ad_{\xi_1}, ad_{\xi_2}, ad_{\xi_3}) \ .
$$

But as in example 1,

$$
ad_{\xi_j} = \begin{cases} \begin{pmatrix} 0 & 0 \\ 0 & L_j \end{pmatrix} = 0 \oplus L_j & \text{if } j = 1, 2, 3 \\[4mm] \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0 & \text{if } j = 0 \end{cases} ,
$$

where

$$
L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \ L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \ L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

is the basis of the Lie algebra $\mathbf{so}\,(3)$ of the special orthogonal group $\mathbb{SO}\,(3)$ given in Appendix B on page 45.

Let

$$
\{\partial/\partial x_{jk} \mid j, k = 0, 1, 2, 3\}
$$

denote the basis of $\mathbf{Vec}\,(\mathbf{u}\,(2^2))$ induced by the chart

$$
\mathbf{u}\,(2^2) \quad \xrightarrow{\ \pi\ } \quad \mathbb{R}^{16}
$$

$$
i\rho = \sum_{i,j=0}^{3} x_{ij}\xi_{ij} \quad \longmapsto \quad (x_{00}, x_{0*}, x_{10}, x_{1*}, x_{20}, x_{2*}, x_{30}, x_{3*})
$$

where

$$
(x_{00}, x_{0*}, x_{10}, x_{1*}, x_{20}, x_{2*}, x_{30}, x_{3*})
$$

$$
= (x_{00}, \quad x_{01}, x_{02}, x_{03}, \quad x_{10}, \quad x_{11}, x_{12}, x_{13}, \quad x_{20}, \quad x_{21}, x_{22}, x_{23}, \quad x_{30}, \quad x_{31}, x_{32}, x_{33})
$$

In other words, for each pair (j, k), $\partial/\partial x_{jk}$ denotes the vector field on $\mathbf{u}\left(2^2\right)$ defined at each point $i\rho$ as the tangent vector to the curve

$$\pi^{-1}\left(x_{00}, \dots, x_{jk} + t, \dots, x_{33}\right) = i\rho + t\xi_{jk}$$

at $t = 0$.

In terms of the above chart, $\Omega\left(v\right)\left(i\rho\right)$ can be written as

$$\left(x_{00}, x_{0*}, x_{10}, x_{1*}, x_{20}, x_{2*}, x_{30}, x_{3*}\right) \cdot \left[I_4 \otimes \left(0 \oplus a \cdot L\right) + \left(0 \oplus b \cdot L\right) \otimes I_4\right] \cdot \begin{pmatrix} \partial/\partial x_{00} \\ \partial/\partial x_{0*} \\ \partial/\partial x_{10} \\ \partial/\partial x_{1*} \\ \partial/\partial x_{20} \\ \partial/\partial x_{2*} \\ \partial/\partial x_{30} \\ \partial/\partial x_{3*} \end{pmatrix},$$

which simplifies to

$$\Omega\left(v\right)\left(i\rho\right) = \sum_{q=0}^{3} \left(a \cdot x_{q*} \times \frac{\partial}{\partial x_{q*}} + b \cdot x_{*q} \times \frac{\partial}{\partial x_{*q}}\right),$$

where '$\times$' denotes the vector cross product[12].

We can now achieve objective 1.

Objective 1. *Given an arbitrary density operator $i\rho$ in $\mathbf{u}\left(2\right)$, find the dimension of an arbitrary entanglement class $[i\rho]_E$.*

From the above discussion, it follows that the image $\operatorname{Im}\left(\Omega\right)$ of the infinitesimal action Ω is spanned by the six vector fields

---

[12]The vector cross product is computed according to the right-hand rule.

$$\begin{cases} \Omega\left(\xi_{01}\right) = \sum_{q=0}^{3}\left(x_{q2}\frac{\partial}{\partial x_{q3}} - x_{q3}\frac{\partial}{\partial x_{q2}}\right) \\[2ex] \Omega\left(\xi_{02}\right) = \sum_{q=0}^{3}\left(x_{q3}\frac{\partial}{\partial x_{q1}} - x_{q1}\frac{\partial}{\partial x_{q3}}\right) \\[2ex] \Omega\left(\xi_{03}\right) = \sum_{q=0}^{3}\left(x_{q1}\frac{\partial}{\partial x_{q2}} - x_{q2}\frac{\partial}{\partial x_{q1}}\right) \end{cases}$$

$$\begin{cases} \Omega\left(\xi_{10}\right) = \sum_{q=0}^{3}\left(x_{2q}\frac{\partial}{\partial x_{3q}} - x_{3q}\frac{\partial}{\partial x_{2q}}\right) \\[2ex] \Omega\left(\xi_{20}\right) = \sum_{q=0}^{3}\left(x_{3q}\frac{\partial}{\partial x_{1q}} - x_{1q}\frac{\partial}{\partial x_{3q}}\right) \\[2ex] \Omega\left(\xi_{30}\right) = \sum_{q=0}^{3}\left(x_{1q}\frac{\partial}{\partial x_{2q}} - x_{2q}\frac{\partial}{\partial x_{1q}}\right) \end{cases}$$

In particular, the tangent space $T_{i\rho}\left([i\rho]_E\right) = \left.(\operatorname{Im}\Omega)\right|_{i\rho}$ to the entanglement class $[i\rho]_E$ at the point $i\rho$ is spanned by

$$\left.\Omega\left(\xi_{01}\right)\right|_{i\rho}, \ \left.\Omega\left(\xi_{02}\right)\right|_{i\rho}, \ \left.\Omega\left(\xi_{03}\right)\right|_{i\rho}, \ \left.\Omega\left(\xi_{10}\right)\right|_{i\rho}, \ \left.\Omega\left(\xi_{20}\right)\right|_{i\rho}, \ \left.\Omega\left(\xi_{30}\right)\right|_{i\rho}$$

We leave it as an exercise for the reader to verify that the above six vector fields are linearly independent almost every where. Thus, it follows that almost all entanglement classes are of dimension six.

However, there are notable exceptions. Consider the Bell state[13], $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right)$. The corresponding density operator $i\rho$ is

$$i\rho = \frac{i}{2}\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} = \left(-\frac{1}{2}\right)\xi_{00} + \left(\frac{1}{2}\right)\xi_{11} + \left(-\frac{1}{2}\right)\xi_{22} + \left(-\frac{1}{2}\right)\xi_{33},$$

where

$$\xi_{jk} = -\frac{i}{2}\sigma_j \otimes \sigma_k.$$

[13]It should be noted that all four 2 qubit Bell states lie in the same entanglement class. It is this fact that makes quantum teleportation possible.

Hence,

$$
x_{jk} = \begin{cases} -\frac{1}{2} & \text{if} \quad j = k = 0, 2, 3 \\[2mm] \frac{1}{2} & \text{if} \quad j = k = 1 \\[2mm] 0 & \text{if} \quad j \neq k \end{cases}
$$

Thus, in this case $\text{Im}\,(\Omega)|_{i\rho}$ is spanned by

$$
\begin{cases} \Omega\,(\xi_{01})|_{i\rho} &=& \frac{1}{2}\left(\frac{\partial}{\partial x_{23}} - \frac{\partial}{\partial x_{32}} \right) \\[3mm] \Omega\,(\xi_{02})|_{i\rho} &=& \frac{1}{2}\left(\frac{\partial}{\partial x_{31}} + \frac{\partial}{\partial x_{13}} \right) \\[3mm] \Omega\,(\xi_{03})|_{i\rho} &=& \frac{1}{2}\left(-\frac{\partial}{\partial x_{12}} - \frac{\partial}{\partial x_{21}} \right) \end{cases}
$$

$$
\begin{cases} \Omega\,(\xi_{10})|_{i\rho} &=& \frac{1}{2}\left(\frac{\partial}{\partial x_{32}} - \frac{\partial}{\partial x_{23}} \right) \\[3mm] \Omega\,(\xi_{20})|_{i\rho} &=& \frac{1}{2}\left(\frac{\partial}{\partial x_{13}} + \frac{\partial}{\partial x_{31}} \right) \\[3mm] \Omega\,(\xi_{30})|_{i\rho} &=& \frac{1}{2}\left(-\frac{\partial}{\partial x_{21}} - \frac{\partial}{\partial x_{12}} \right) \end{cases}
$$

Hence,

$$
Dim\,[i\rho_{Bell}]_E = Dim\left[(\text{Im}\,\Omega)|_{i\rho_{Bell}} \right] = 3
$$

This only confirms the conventional wisdom that the entanglement class of the Bell states is truly exceptional.

We are now ready for objective 2:

Objective 2. *Given two states $i\rho$ and $i\rho'$, devise a means of determining whether they belong to the same or to different entanglement classes.*

The complete functionally independent set of solutions to the above system of PDEs (hence, a complete set of entanglement invariants) was found by Linden and Popescu in[25]. These invariants are as described below[14]. For further details please refer to [25].

---

[14]We are using a notation different from that found in [25].

Let $i\rho$ be an arbitrary element of the Lie algebra $\mathbf{u}\left(2^2\right)$. Then in terms of the earlier described chart π,

$$i\rho = \sum_{j,k=0}^{3} x_{jk}\xi_{jk} \ ,$$

where $\{\xi_{jk}\}$ denotes the basis of $\mathbf{u}\left(2^2\right)$ described earlier.

We will change our notation slightly. Let x_{**} denote the 3×3 matrix

$$x_{**} = \left(x_{jk}\right)_{j,k=1,2,3} \ ,$$

and let x_{0*} and x_{*0} denote the vectors

$$\begin{cases} x_{0*} &= (x_{01}, x_{02}, x_{02}) \\[2mm] x_{*0} &= (x_{10}, x_{20}, x_{30}) \end{cases}$$

Finally, let Z denote the matrix

$$Z = x_{**}x_{**}^{T} \ ,$$

where the superscript 'T' denotes the transpose. Then the nine algebraically independent polynomial functions listed in the table

$Tr\left(Z\right)$	$Tr\left(Z^2\right)$	$\det\left(x_{**}\right)$
$x_{0*}x_{0*}^{T}$	$x_{0*}Zx_{0*}^{T}$	$x_{0*}Z^2x_{0*}^{T}$
$x_{0*}x_{**}x_{*0}^{T}$	$x_{0*}Zx_{**}x_{*0}^{T}$	$x_{0*}Z^2x_{**}x_{*0}^{T}$

form a basic set of entanglement invariants.

But the above nine entanglement invariants do not form a complete set of entanglement invariants! A tenth polynomial function

$$x_{0*} \cdot \left(Zx_{0*}^{T}\right) \times \left(Z^2x_{0*}^{T}\right)$$

is needed to form a complete system of entanglement invariants. Although this tenth entanglement invariant is algebraically dependent on the above nine entanglement invariants, it is still needed to determine the sign of the components of $i\rho$.

16. Example n. The entanglement classes of n qubits, $n > 2$

For n qubits $(n > 2)$, the same methods lead to the following formula for the infinitesimal action

$$
\Omega\left(v\right)\left(i\rho\right) = \sum_{q_1,q_2\cdots q_{n-1}=0}^{3}\sum_{k=1}^{n} a^{(k)} \cdot x_{q_1 q_2 \cdots q_{k-1} * q_{k+1} \cdots q_{n-1}} \times \frac{\partial}{\partial x_{q_1 q_2 \cdots q_{k-1} * q_{k+1} \cdots q_{n-1}}}
$$

where $v \in \ell\left(2^n\right)$ and $i\rho \in \mathbf{u}\left(2^n\right)$ are given by

$$
\begin{cases}
v &= \displaystyle\sum_{k=1}^{n} a^{(k)} \cdot \xi_{\underbrace{00\cdots0*0\cdots0}_{*\ \text{in}\ k\text{-th position}}} \\[3em]
i\rho &= \displaystyle\sum_{r_1,r_2,\cdots,r_n=0}^{3} x_{r_1 r_2 \cdots r_n} \xi_{r_1 r_2 \cdots r_n}
\end{cases}
$$

We will leave the solution to the corresponding system of PDEs to future papers.

17. Conclusion

There is much more that could be said about quantum entanglement. This paper presents only a small part of the big picture. But hopefully this paper will provide the reader with some insight into this rapidly growing research field. Since this paper was written, research in quantum entanglement has literally had an explosive expansion, and even now continues to do so. We refer the reader to the references at the end of this paper, which represent only a few of the many papers in this rapidly expanding field.

18. Appendix A. Some Fundamental Concepts from the Theory of Differential Manifolds

18.1. Differential manifolds, tangent bundles, and vector fields.

Definition 12. *A topological space M^m is an* **m-dimensional manifold** *if it is locally homeomorphic to $\mathbb{R}^m$, i.e., if there exists an open cover $\mathcal{W} = \{W_\alpha\}$ of M^m such that for each $W_\alpha \in \mathcal{W}$, there is associated a homeomorphism*

$$
W_\alpha \xrightarrow{\varphi_\alpha} \mathbb{R}^m
$$
$$
x \longmapsto (x_1, x_2, \ldots, x_m)
$$

which maps W_α onto an open subset of $\mathbb{R}^m$.   We call

$$(\varphi_\alpha, W_\alpha)$$

*a **chart on** M^m, and*

$$\Phi = \{\ (\varphi_\alpha, W_\alpha)\ \}$$

*an **Atlas** on M^m.*

*An Atlas is said to be **smooth** (C^∞), if whenever*

$$\varphi_\beta \varphi_\alpha^{-1} : \varphi_\alpha\,(W_\alpha \cap W_\beta) \longrightarrow \varphi_\beta\,(W_\alpha \cap W_\beta)$$

*is defined, is a smooth (C^∞) map of $\varphi_\alpha\,(W_\alpha \cap W_\beta) \subseteq \mathbb{R}^n$ into $\varphi_\beta\,(W_\alpha \cap W_\beta) \subseteq \mathbb{R}^n$.   A **smooth** (C^∞) **manifold** is a topological manifold with a smooth atlas.*
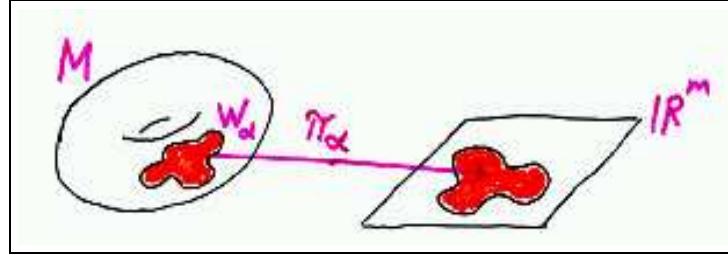

**Figure 6.   A chart π_α: M $\longrightarrow \mathbb{R}^4$ on a manifold M.**
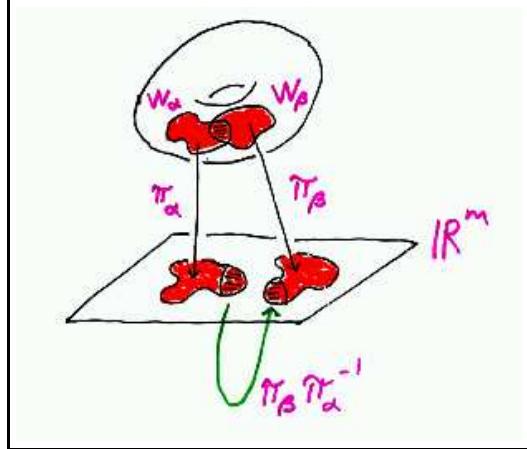
Figure 7.   An atlas is smooth if every $\pi_\alpha \pi_\beta^{-1}$ is smooth when defined.

Definition 13. *Let M and N be smooth manifolds.   Then a map*

$$f : M \longrightarrow N$$

is said to be smooth if for every $x \in M$ there exist charts $(\varphi_\alpha, W_\alpha)$ of M and (ψ_β, V_β) of N containing x and $f(x)$ respectively such that

$$\psi_\beta f \varphi_\alpha^{-1} : \varphi_\alpha(W_\alpha) \longrightarrow \varphi_\beta(V_\beta)$$

is smooth.

Definition 14. *Let x be an element of a smooth manifold M, and let $\gamma_1(t)$ and $\gamma_2(t)$ be smooth curves in M which pass through x, i.e., such that there exists $t_1, t_2 \in \mathbb{R}$ for which*

$$\gamma_1(t_1) = x = \gamma_2(t_2)$$

*Then γ_1 and γ_2 are said to be **tangentially equivalent** at x, written*

$$\gamma_1 \underset{x}{\sim} \gamma_2 \ ,$$

if they are tangent at the point x, i.e., if there is a chart $(\varphi_\alpha, W_\alpha)$ on M containing x such that

$$\frac{d}{dt} (\varphi_\alpha \circ \gamma_1)(t)\,|_{t=t_1} = \frac{d}{dt} (\varphi_\alpha \circ \gamma_2)(t)\,|_{t=t_2}$$

Remark 5. *It can easily be shown that the relation $\underset{x}{\sim}$ is independent of the chart selected.*

Definition 15. *A **tangent vector** (x, v) (also written simply as v) to M at x is a tangential equivalence class at x. The tangent space of M^n at x, denoted by $T_x M^n$, is the set of tangent vectors to M at x. $T_x M$ can be shown to be an n-dimensional vector space.*

Let

$$TM = \bigcup_{x \in M} T_x M \ ,$$

and let π be the map

$$TM \xrightarrow{\ \pi\ } M$$
$$(x, v) \longmapsto x$$

If $\varphi_\alpha : W_\alpha \longrightarrow \mathbb{R}^m$ is a chart on M, then

$$\varphi_\alpha \pi : \pi^{-1} W_\alpha \longrightarrow \mathbb{R}^m$$

*can be shown to be a chart on TM. In this way, TM becomes a smooth manifold and π becomes a smooth map. TM together with the map π is called the **tangent bundle** of M.*

Definition 16. *A **vector field** v on a smooth manifold M^m is a smooth map*

$$v : M^m \longrightarrow TM^m$$

Let $\mathbf{Vec}(M^m)$ be the set of all vector fields on the smooth manifold M^m. This is easily seen to be a vector space where, for example, the sum $u + v$ of two vector fields is defined by

$$(u + v)|_x = u|_x + v|_x$$

for all $x \in M^m$.

We will now consider the charts of the tangent bundle TM in a more explicit way.

Let

$$W_\alpha \xrightarrow{\varphi_\alpha} \mathbb{R}^m$$

$$x \longmapsto (x_1, x_2, \dots, x_m)$$

be a chart on the smooth manifold M^m, and let a be an arbitrary point in U_α. Thus,

$$\varphi_\alpha(a) = (a_1, a_2, \dots, a_m) \ .$$

For each j $(j = 1, 2, \dots, m)$ consider the smooth curve

$$\gamma_j(t) = \varphi_\alpha^{-1}(a_1, a_2, \dots, a_j + t, \dots, a_m)$$

in U_α which passes through the point a at time $t = 0$. Then for each such j, let

$$\left. \frac{\partial}{\partial x_j} \right|_a \in T_a M$$

denote the tangent vector to the curve γ_j at a. It can be shown that

$$\left. \frac{\partial}{\partial x_1} \right|_a, \ \left. \frac{\partial}{\partial x_2} \right|_a, \ \dots, \ \left. \frac{\partial}{\partial x_m} \right|_a$$

is a vector space basis of the tangent space $T_a M$.

Moreover, since this construction is respect to an arbitrary point a in W_α, it can be shown that we have actually constructed for each j a smooth vector field

$$\frac{\partial}{\partial x_j} \in \mathbf{Vec}(TW_\alpha) \subseteq \mathbf{Vec}(TM)$$

In fact, it can be shown that

$$\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ \dots, \ \frac{\partial}{\partial x_m}$$

is a basis of $\mathbf{Vec}(TW_\alpha)$, and hence a local basis of $\mathbf{Vec}(TM)$.

We can now express each chart $\left(\varphi_\alpha \pi, \pi^{-1} W_\alpha\right)$ explicitly as:

$$\pi^{-1} W_\alpha \xrightarrow{\varphi_\alpha \pi} \mathbb{R}^{2m}$$

$$(x, \mu_1 \frac{\partial}{\partial x_1} + \mu_2 \frac{\partial}{\partial x_2} + \ldots + \mu_m \frac{\partial}{\partial x_m}) \longmapsto (x_1, x_2, \ldots, x_m, \ \mu_1, \mu_2, \ldots, \mu_m)$$

where μ_j's on the left denote functions of $x \in M$, and where μ_j's on the right denote functions of $(x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$.

Definition 17. *Let M and N be smooth manifolds, and let $f : M \longrightarrow N$ be a smooth map, and let a be an arbitrary point of M. We define a vector space morphism*

$$df|_a : T_a M \longrightarrow T_{f(a)} N$$

as follows:

For each $v \in T_a M$, there is a representative smooth curve $\gamma_v(t)$ in M which passes through the point a and which has v as its tangent vector at the point a. It follows that $f \circ \gamma_v(t)$ is a smooth curve in N passing through the point $f(a)$. We define

$$df|_a (v) \in T_{f(a)} N$$

as the tangent vector to $f \circ \gamma_v(t)$ at the point $f(a)$. It is then a simple exercise to show that $df|_a$ is a vector space morphism.

Since a was an arbitrary point of M, this leads to the definition of a smooth map $df : TM \longrightarrow TN$, called the differential of f, such that the following diagram is commutative:

$$
\begin{array}{ccc}
TM & \xrightarrow{df} & TN \\
\downarrow & & \downarrow \\
M & \xrightarrow{f} & N
\end{array}
$$

Remark 6. *In local coordinates, df maps the tangent vector*

$$v|_x = \sum_{i=1}^{m} \mu_i \frac{\partial}{\partial x_i}$$

to the tangent vector

$$df(v|_x) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} \mu_i \frac{\partial f_j}{\partial x_i} \right) \frac{\partial}{\partial y_j} .$$

Thus, the matrix expression of the linear transformation df is just the Jacobian matrix

$$\left(\frac{\partial f_j}{\partial x_i}(x) \right)_{m \times n} .$$

18.2. **Exponentiation of vector fields.**

Definition 18. *Let M be a smooth manifold, and let $v \in \mathbf{Vec}(M)$ be a smooth vector field on M. A curve $\gamma(t)$ in M is said to be an **integral curve** of v if $v|_{\gamma(t)}$ is the tangent vector to $\gamma(t)$ for each t for which $\gamma(t)$ is defined.*

In terms of local coordinates, an integral curve $\gamma(t)$ of a smooth vector field

$$v(x) = \sum_{i=1}^{n} \mu_i(x_1, x_2, \dots , x_m) \frac{\partial}{\partial x_i}$$

is a solution to the system of ordinary differential equations

$$\frac{dx_i}{dt} = \mu_i(x_1, x_2, \dots , x_m) \ , \ i = 1, 2, \dots , m$$

Since v is smooth, its coefficients $\mu^i(x_1, x_2, \dots , x_m)$ are smooth functions. Consequently, it follows from the standard existence and uniqueness theorems for systems of ordinary differential equations that there exists a unique solution for each set of initial conditions.

Thus, for each x in M, there exists a unique maximal integral curve $\gamma_v(t, x)$ passing through x at time $t = 0$, and call $\gamma_v(t, x)$ the **flow** generated by the vector field v. We call v the **infinitesimal generator** of the flow. It can be easily shown that

$$\gamma_v(t, \gamma_v(s, x)) = \gamma_v(t + s, x) \ \ .$$

Hence, we are justified in adopting the following suggestive notation:

$$e^{tv} x$$

for the flow $\gamma_v(t, x)$.

In terms of our new notation, the properties of the flow can be expressed as

1) $e^{sv} e^{tv} x = e^{(s+t)v} x$
2) $e^{0 \cdot v} x = x$
3) $\frac{d}{dt} \left(e^{tv} x \right) = v|_{e^{tv} x} \ \ .$

18.3. **Vector fields viewed as directional derivatives.** We now show how vector fields can be viewed as partial differential operators.

Definition 19. *A **derivation** D on an algebra $\mathcal{A}$ is a map*

$$D : \mathcal{A} \to \mathcal{A}$$

such that

1) *(Linearity)* $D(\alpha f + \beta g) = \alpha D f + \beta D g$

2) *(Leibnitz Rule)* $D\left(fg\right) = \left(Df\right)g + f\left(Dg\right)$

Definition 20. *A **Lie algebra** $\mathbb{A}$ is a vector space together with a binary operation*

$$[-,-] : \mathbb{A} \times \mathbb{A} \to \mathbb{A} \ ,$$

*called a **Lie bracket** for $\mathbb{A}$, such that*

1) *(Bilinearity)*

$$[\lambda_1 a_1 + \lambda_2 a_2, \ b] \ = \ \lambda_1\left[a_1, b\right] + \lambda_2\left[a_2, b\right]$$

$$[a, \ \lambda_1 b_1 + \lambda_2 b_2] \ = \ \lambda_1\left[a, b_1\right] + \lambda_2\left[a, b_2\right]$$

2) *(Skew-Symmetry)*

$$[a, b] = -\left[b, a\right]$$

3) *(Jacobi Identity)*

$$[a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0$$

Proposition 3. *The set of derivations $Der(\mathcal{A})$ on an algebra $\mathcal{A}$ is a Lie algebra with Lie bracket given by:*

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$$

Let $\mathbf{C}^\infty(M)$ denote the **algebra of real valued functions** on the smooth manifold M. Then, it follows that $\mathbf{Der}(\mathbf{C}^\infty(M))$ is a Lie algebra.

We will now show how to identify the elements of $\mathbf{Vec}(M)$ with derivations in $\mathbf{Der}(\mathbf{C}^\infty(M))$, and thereby show that $\mathbf{Vec}(M)$ is more than a vector space. It is actually a Lie algebra.

Each smooth vector field v on M can be thought of as a directional derivative in the direction v as follows: Let $v \in \mathbf{Vec}(M)$ and let $f \in \mathbf{C}^\infty(M)$. Define $v(f)$ as:

$$v(f)|_x = \left.\frac{d}{dt} f\left(e^{tv}x\right)\right|_{t=0}$$

Thus, we have:

Proposition 4. $\mathbf{Vec}(M)$ *is a Lie algebra of derivations on the algebra $\mathbf{C}^\infty(M)$.*

It is enlightening, to view the above in terms of local coordinates. From this perspective,

$$v = \sum_{i=1}^{m} \mu_i(x) \frac{\partial}{\partial x_i} \ .$$

Thus, if we use the chain rule and the fact that

$$\frac{d}{dt}\left(e^{tv}x\right) = v|_{e^{tv}x} \ ,$$

we have

$$\frac{d}{dt} f\left(e^{tv}x\right) = \sum_{i-1}^{m} \xi^i\left(e^{tv}x\right) \frac{\partial f}{\partial x^i}\left(e^{tv}x\right) = \left(\sum_{i=1}^{m} \xi^i \frac{\partial}{\partial x^i}\right)(f)\Bigg|_{e^{tv}x} \ .$$

Hence,

$$v = \sum_{i=1}^{m} \mu_i \frac{\partial}{\partial x_i} \in Vec(U_\alpha) \subseteq Vec(M)$$

acts as a first order partial differential operator, thereby justifying the notation.

So viewing v as a first order partial differential operator, we can write

$$\frac{d}{dt} f\left(e^{tv}x\right) = v(f)|_{e^{tv}x} \ ,$$

and, in particular,

$$\frac{d}{dt} f\left(e^{tv}x\right)\Bigg|_{t=0} = v(f)(x) \ ,$$

where $v(f)(x)$ now denotes (locally) $\left(\sum_{i=1}^{m} \mu_i \frac{\partial}{\partial x_i}\right) f$ evaluated at x.

19. Appendix B. Some Fundamental Concepts from the Theory of Lie Groups

19.1. Lie groups.

Definition 21. *A **Lie group** $\mathbb{G}$ is a group which is a smooth manifold whose differential structure is compatible with the group operations, i.e., such that*

1) *The multiplication map of $\mathbb{G}$*

$$\mathbb{G} \times \mathbb{G} \longrightarrow \mathbb{G}$$
$$(g_1, g_2) \longmapsto g_1 g_2$$

   and,

2) *The inverse map of G*

$$\mathbb{G} \longrightarrow \mathbb{G}$$
$$g \longmapsto g^{-1}$$

are smooth functions.

*A closed subgroup $\mathbb{H}$ of $\mathbb{G}$ can be shown to be a subgroup, and hence is called a **Lie subgroup** of $\mathbb{G}$.*

Definition 22. *A **one parameter subgroup** of a Lie group $\mathbb{G}$ is a smooth morphism from the additive Lie group of reals $\mathbb{R}, +$ to the group $\mathbb{G}$.*

19.2. **Some examples of Lie groups.** Let V denote an n-dimensional vector space over the real numbers $\mathbb{R}$ with the standard vector inner product which we denote by $\langle\,,\,\rangle$.

- $\mathbb{GL}(n, \mathbb{R})$ The **real general linear group** of all automorphisms of the vector space V. This can be identified with the group of all nonsingular $n \times n$ matrices over the reals.
- $\mathbb{O}(n)$ The **real orthogonal group** is the group of all automorphisms which preserve the inner product $\langle\,,\,\rangle$. This can be identified with the group of orthogonal matrices , i.e., matrices A of the form

$$A^T = A^{-1}$$

  where the superscript "T" denotes the matrix transpose.
- $\mathbb{SL}(n, \mathbb{R})$ The **real special linear group** is the group of all real $n \times n$ matrices of determinant 1. $SL(n, \mathbb{R})$ is the group of all rigid motions in hyperbolic n-space.
- $\mathbb{SO}(n) = \mathbb{O}(n) \cap \mathbb{SL}(n, \mathbb{R})$ The **special orthogonal group** is the group of all orthogonal real $n \times n$ matrices of determinant 1. This group can be identified with the group of all rotations in $\mathbb{R}^n$ about a fixed point such as the origin.

Let W denote an n-dimensional vector space over the complex numbers $\mathbb{C}$ with the standard sesquilinear inner product which we also denote by $\langle\,,\,\rangle$.

- $\mathbb{GL}(n, \mathbb{C})$ The **complex general linear group** of all automorphisms of the vector space W. This can be identified with the group of all nonsingular $n \times n$ matrices over the complexes.
- $\mathbb{SL}(n, \mathbb{C})$ The **complex special linear group** is the group of all complex $n \times n$ matrices of determinant 1.

- $\mathbb{U}(n)$ The **unitary group** is the group of all $n \times n$ unitary matrices over the complex numbers $\mathbb{C}$, i.e., all $n \times n$ complex matrices A such that

$$A^\dagger = A^{-1}$$

  where $A^\dagger$ denotes the conjugate transpose.
- $\mathbb{SU}(n) = \mathbb{U}(n) \cap \mathbb{SL}(n, \mathbb{C})$ The special unitary group is the group of all unitary matrices of determinant 1.

19.3. **The Lie algebra of a Lie group.**

Definition 23. *Let $\mathbb{G}$ be a Lie group. For each element $h \in \mathbb{G}$, we define the **right multiplication map**, written R_h, as*

$$\mathbb{G} \xrightarrow{R_h} \mathbb{G}$$

$$g \longmapsto gh$$

The map R_h is an autodiffeomorphism of $\mathbb{G}$. We let

$$dR_h : T\mathbb{G} \longrightarrow T\mathbb{G}$$

denote the corresponding differential of this diffeomorphism.

*Finally, a vector field $v \in \mathbf{Vec}(\mathbb{G})$ is said to be **right invariant** if*

$$(dR_h)\left(v|_g\right) = v|_{gh}$$

Definition 24. *Let $\mathbf{Vec}_R(\mathbb{G})$ denote the set of right invariant vector fields on $\mathbb{G}$. Then $\mathbf{Vec}_R(\mathbb{G})$ as a subset of the Lie algebra $\mathbf{Vec}(\mathbb{G})$ inherits the structure of a Lie algebra. We call $\mathbf{Vec}_R(\mathbb{G})$ the **Lie algebra** of the Lie group $\mathbb{G}$.*

Let I denote the identity element of the Lie group $\mathbb{G}$. Since a right invariant vector field $v \in \mathbf{Vec}_R(\mathbb{G})$ is completely determined by its restriction to the tangent space $T_I\mathbb{G}$ via

$$v|_g = (dR_g)\left(v|_I\right)\ ,$$

we can, and do, identify the Lie algebra $\mathbf{Vec}_R(\mathbb{G})$ with the tangent space $T_I\mathbb{G}$, i.e.,

$$\mathbf{Vec}_R(\mathbb{G}) = T_I\mathbb{G}\ .$$

The tangent bundle $T\mathbb{G}$ of a Lie group $\mathbb{G}$ is trivial. For it can be shown that $T\mathbb{G}$ is bundle isomorphic to $\mathbb{G} \times T_I\mathbb{G}$. However, there is some additional and useful structure induced on the Lie algebra $Vec_R(\mathbb{G}) = T_I\mathbb{G}$ by the Lie group structure of $\mathbb{G}$, i.e., the exponential map.

Definition 25. *We define the **exponential map** exp from the Lie algebra* $\mathbf{Vec}_R(\mathbb{G})$ *to the Lie group G as*

$$\mathbf{Vec}_R(\mathbb{G}) \xrightarrow{\exp} \mathbb{G}$$

$$v \longmapsto \left(e^{vt}\right) I\big|_{t=1}$$

In other words, we simply follow the flow $\gamma_v(t,g) = e^{vt}g$ from the identity I to the point $e^v I$ in $\mathbb{G}$.

It can be shown that the exponential map

$$\exp : \mathbf{Vec}_R(\mathbb{G}) \longrightarrow \mathbb{G}$$

is a local diffeomorphism. It also follows that, for each $v \in \mathbf{Vec}_R(\mathbb{G})$, $\exp(tv)$ is a one parameter subgroup of G. In fact, all one parameter subgroups are of this form.

19.4. Some examples of Lie algebras.

19.4.1. *Example: The Lie algebra $\mathbf{u}(N)$ of the unitary group $\mathbb{U}(N)$.* In this case, $\mathbf{u}(N)$ is the Lie algebra of all $N \times N$ skew Hermitian[15] matrices of $\mathbb{C}$. This can be seen as follows:

The Lie algebra $\mathbf{u}(N)$ is the tangent space $T^I \mathbb{U}(N)$ to $\mathbb{U}(N)$ at the $N \times N$ identity matrix I. Hence, $\mathbf{u}(N)$ consists of all tangent vectors $\overset{\bullet}{U}(0) = \frac{d}{dt}U(t)\big|_{t=0}$ of all curves $U(t)$ in $\mathbb{U}(N)$ which pass through I at $t = 0$, i.e., which satisfy $U(0) = I$.

Since $U(t)$ is unitary, i.e., since

$$U(t)\overline{U}(t)^T = I,$$

we find by differentiating the above formula that

$$\overset{\bullet}{U}(t)\overline{U}(t)^T + U(t)\overline{\overset{\bullet}{U}}(t)^T = 0 \ .$$

Setting $t = 0$, we have

$$\overline{\overset{\bullet}{U}}(0)^T = -\overset{\bullet}{U}(0) \ \ .$$

Thus all matrices in $\mathbf{u}(N)$ are skew Hermitian.

Let M be an arbitrary skew $N \times N$ Hermitian matrix. Then

$$U(t) = \exp(tM)$$

---

[15]A square matrix M is skew Hermitian if $\overline{M}^T = -M$.

is a curve in $\mathbb{U}(N)$ which passes through I at $t = 0$ for which $\overset{\bullet}{U}(0) = M$. Hence, $\mathbf{u}(N)$ is the Lie algebra of all $N \times N$ skew Hermitian matrices over $\mathbb{C}$.

Let

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

denote the Pauli spin matrices, and let

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

denote the 2×2 identity matrix. Then the following is a basis of the Lie algebra $\mathbf{u}(2^n)$

$$\{\xi_{j_1 j_2 \cdots j_n} \mid j_1, j_2, \ldots, j_n = 0, 1, 2, 3\} \;,$$

where

$$\xi_{j_1 j_2 \cdots j_n} = -\frac{i}{2}\sigma_{j_1} \otimes \sigma_{j_2} \otimes \ldots \otimes \sigma_{j_n} \;.$$

Remark 7. *Please note that, although $\mathbf{u}(N)$ is a Lie algebra of complex matrices, it is nonetheless a real Lie algebra. Thus, the above basis $\{\xi_{j_1 j_2 \cdots j_n}\}$ of $\mathbf{u}(2^n)$ is a basis of $\mathbf{u}(2^n)$ over the reals $\mathbb{R}$. But the matrices in $\mathbf{u}(2^n)$ are still matrices of complex numbers!*

19.4.2. *Example: The Lie algebra $\mathbf{su}(N)$ of the special unitary group $\mathbb{SU}(N)$.* The Lie algebra $\mathbf{su}(N)$ for the special unitary group is the same as the Lie algebra of all $N \times N$ traceless skew Hermitian matrices, i.e., of all $N \times N$ skew Hermitian matrices M such that $trace(M) = 0$. A basis of the Lie algebra $\mathbf{su}(2^n)$ is

$$\{\xi_{j_1 j_2 \cdots j_n} \mid j_1, j_2, \ldots, j_n = 0, 1, 2, 3\} - \{\xi_{00 \cdots 0}\} \;.$$

19.4.3. *Example: The Lie algebra $\mathbf{so}(3)$ of the special unitary group $\mathbb{SO}(3)$.* Finally, we should mention that the Lie algebra $\mathbf{so}(3)$ of the special orthogonal group $\mathbb{SO}(3)$ is the Lie algebra of all 3×3 skew symmetric matrices over the reals $\mathbb{R}$. The following three matrices form a basis for $so(3)$

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \;.$$

19.5. **Lie groups as transformation groups on manifolds.**

Definition 26. *Let M be a smooth manifold. Then a **group of transformations acting on** M is a Lie group $\mathbb{G}$ together with a smooth map*

$$\mathbb{G} \times M \longrightarrow M$$
$$(g, x) \longmapsto g \cdot x$$

such that

1) *For all $x \in M$, and for all $g_1, g_2 \in \mathbb{G}$*

$$g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$$

2) *For all $\in M$,*

$$e \cdot x = x \ ,$$

where e denotes the identity of $\mathbb{G}$.
$\mathbb{G}$ is called a transformation group of M.

Definition 27. *Let M be a smooth manifold, and let $\mathbb{G}$ be a Lie group acting on M. Then the action*

$$\mathbb{G} \times M \longrightarrow M$$

*induces an **infinitesimal action***

$$\mathbf{Vec}_R(\mathbb{G}) \xrightarrow{\Psi_G} \mathbf{Vec}(M) \ ,$$

where $\Psi_{\mathbb{G}}(v)|_x$ is the tangent vector to the curve

$$\gamma_v(t, x) = e^{tv} x$$

in M at x, i.e.,

$$\Psi_{\mathbb{G}}(v)|_x = \left. \frac{d}{dt} \left(e^{tv} x \right) \right|_{t=0} \quad .$$

19.6. **The big and little adjoint representations.** Let $\mathbb{G}$ be a Lie algebra, and let $\mathfrak{g}$ denote the corresponding Lie algebra.

For each element $h \in \mathbb{G}$, consider the inner automorphism:

$$\mathbb{G} \xrightarrow{\mathcal{I}_h} \mathbb{G}$$
$$g \longmapsto hgh^{-1}$$

and let

$$T\mathbb{G} \xrightarrow{d\mathcal{I}_h} T\mathbb{G}$$

denote the corresponding differential. We can now define the **big adjoint representation**

$$Ad : \mathbb{G} \longrightarrow Aut(\mathfrak{g})$$

by

$$Ad_h = (d\mathcal{I}_h)|_I$$

where I denotes the identity of G, and where $Aut(\mathfrak{g})$ denotes the group of automorphisms of the Lie algebra $\mathfrak{g}$.

We can now in turn define the **little adjoint representation**

$$ad : \mathfrak{g} \longrightarrow End(\mathfrak{g})$$

of the Lie algebra $\mathfrak{g}$ by

$$ad_v(u) = [u, v] \ ,$$

where $[-, -]$ denotes the Lie bracket, and where $End(\mathfrak{g})$ denotes the ring of endomorphisms of the Lie algebra $\mathfrak{g}$.

As the story goes, $End(\mathfrak{g})$ is actually the Lie algebra of the Lie group $Aut(\mathfrak{g})$, and we have the following commutative diagram

$$
\begin{array}{ccc}
\mathfrak{g} & \xrightarrow{ad} & End(\mathfrak{g}) \\
\exp \downarrow & & \downarrow \exp \\
G & \xrightarrow{Ad} & Aut(\mathfrak{g})
\end{array}
$$

which relates the big and little adjoints. Little adjoint ad is actually the differential $d(Ad)$ restricted to the identity I of the big adjoint Ad.

Perhaps the following example would be of help:

Example 2. *Let $\mathbb{G}$ be the special unitary group $\mathbb{SU}(2)$. Let $su(2)$ denote its Lie algebra. Then $Aut(\mathbb{G})$ is the special orthogonal group $\mathbb{SO}(3)$ and $End(\mathfrak{g})$ is the Lie algebra $so(3)$ of $\mathbb{SO}(3)$. Thus, we have the familiar commutative diagram*

$$
\begin{array}{ccc}
su(2) & \xrightarrow{ad} & so(3) \\
\exp \downarrow & & \downarrow \exp \\
\mathbb{SU}(2) & \xrightarrow{Ad} & \mathbb{SO}(3)
\end{array}
$$

used in quantum mechanics and in quantum computation.

Remark 8. *The reader should verify that*

$$ad_{\xi_j} = L_j$$

19.7. **The orbits of transformation Lie group actions.** Finally, we should remark that the entanglement classes defined previously in this paper are nothing more than the orbits of a group action. For completeness, we give the definition below:

Definition 28. *A subset $\mathcal{O}$ of the smooth manifold M is an **orbit** of the action of the group $\mathbb{G}$ on M provided*

1) *$x \in \mathcal{O} \implies g \cdot x \in \mathcal{O}$ for all $g \in \mathbb{G}$, and*
2) *If S is a non-empty subset of $\mathcal{O}$ which satisfies condition 1) above, then $S = \mathcal{O}$.*

In other words, an orbit is a minimal nonempty invariant subset of M.

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